Hypothesis Testing and Fisher's Exact Test

CS 3130 / ECE 3530: Probability and Statistics for Engineers

April 15, 2025

The Lady Tasting Tea

- Introduced by R.A. Fisher in 1935.
- A lady claimed she could tell whether milk or tea was added first.
- Fisher designed an experiment with 8 cups (4 milk-first, 4 tea-first).
- Lady identifies which were milk-first.
- Question: Can she actually tell or is she guessing?

Null Hypothesis

- **Null Hypothesis ($\mathbf{H_0}$)**: The lady is guessing.
- We try to find evidence to reject H_0.
- Under H_0: she randomly chooses 4 out of 8 cups.
- Total combinations: $\binom{8}{4} = 70$

Contingency Table (Perfect Score)

	Milk First	**Tea First**
Milk First	4	0
Tea First	0	4

$P(\text{all correct}) = \frac{1}{70} \approx 0.014$
If she is guessing, then the probability of getting all correct is too low! In practice, it is **NOT** possible to succeed in just one trial. So we reject the Null hypothesis.

Summary: Hypothesis Test Procedure

1. Define the null hypothesis (H_0): uninteresting or default explanation
2. Assume H_0 is true, derive probability model.
3. Compute probability of observed or more extreme outcomes.

Key intuition: **if under the null hypothesis, the observed outcomes lead to a rare event. Then we should reject the null hypothesis, because a rare event should not be observed in one experiment!**

Formal Procedure

Step 1: Formulate Hypotheses

- Null hypothesis H_0: e.g., random guessing
- Alternative hypothesis H_1: skill.

Step 2: Design Experiment

- Random sample, define test statistic T.
- Choose significance level α.

Test Statistic & Critical Value

- Example: sample mean $\bar{X}_n$
- If $X_i \sim N(\mu, \sigma^2)$, then $Z = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$
- Critical value: $t_\alpha = \mu + z_\alpha \frac{\sigma}{\sqrt{n}}$

Step 3: Run the Experiment

- Collect data $x_1, \ldots, x_n$
- Compute test statistic t
- If $t > t_\alpha$, reject H_0
- Otherwise, do not reject H_0

p-value

- p-value: probability of observing $T \geq t$ under H_0
- If $p < \alpha$, reject H_0

Unknown Variance Case

- If variance σ^2 is unknown:
- Use $T = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$
- Follows t-distribution with $n-1$ degrees of freedom
- Critical value t_α from $t(n-1)$ distribution

R Example

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p = 1 - pt(t, df = n - 1)
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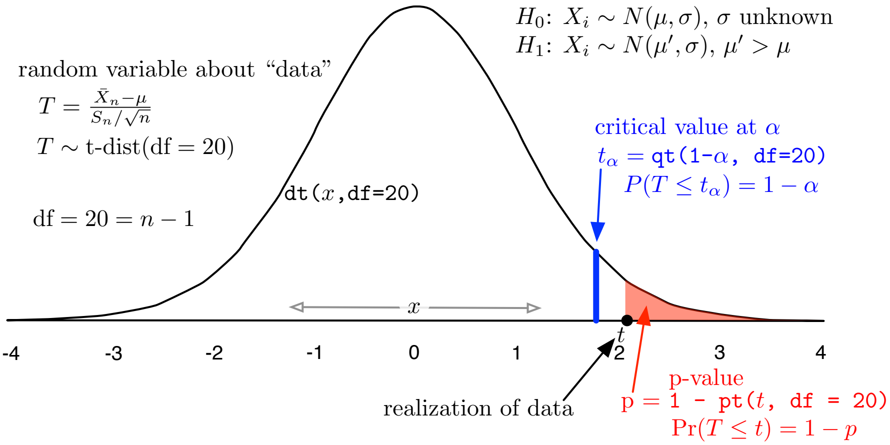
- Computes p-value in R for t-distribution

Further Reading

- Ronald Fisher
- Fisher's Exact Test
- Hypergeometric Distribution

Diagram



H_0: $X_i \sim N(\mu, \sigma)$, σ unknown
H_1: $X_i \sim N(\mu', \sigma)$, $\mu' > \mu$
random variable about "data"
$T = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$
$T \sim$ t-dist(df $= 20$)
df $= 20 = n - 1$
dt(x,df=20)
critical value at α
$t_\alpha =$ qt(1-α, df=20)
$P(T \leq t_\alpha) = 1 - \alpha$
x
t
-4
-3
-2
-1
0
1
2
3
4
realization of data
p-value
p = 1 - pt(t, df = 20)
$\Pr(T \leq t) = 1 - p$