Covariance, Correlation, Bivariate Gaussians

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Expectation of Joint Random Variables

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For continuous:

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx\, dy$$

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Why? Follows from rearranging the summation and marginalizing.

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Alternative Definition:

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Exercise: Prove these two formulas for $\text{Cov}(X, Y)$ are equal.

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Notation:

$$\sigma_X^2 = \text{Var}(X), \quad \sigma_{X,Y} = \text{Cov}(X,Y)$$

Covariance and Independence

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Tricky Fact: $\mathrm{Cov}(X, Y) = 0$ does **not** imply independence!

Correlation

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Solution: **Correlation**

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Interpretation: Correlation is scale-free measure of dependence

Bivariate Gaussian Distribution

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$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left(-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right)$$

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Correlated Case:

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot \exp\Bigg(-\frac{1}{2(1-\rho^2)}$$
$$\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\Bigg)$$

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Variance of Addition

$$\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X, Y)$$