Joint Probability and Independence for Continuous RV's

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Joint Probability and Independence for Contir

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• For two continuous random variables X and Y, consider:

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• The joint pdf f(x, y) satisfies:

1
$$f(x,y) \ge 0$$

2 $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1$
3 $P(a \le X \le b, c \le Y \le d) = \int_{c}^{d} \int_{a}^{b} f(x,y) \, dx \, dy$

Double Integrals

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- Procedure:

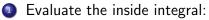
Evaluate the inside integral:

$$\int_a^b f(x,y)\,dx = F(y)$$

(treat y as constant)

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2 Evaluate the outside integral:

$$\int_{c}^{d} F(y) \, dy$$

Example: Double Integral Calculation

Joint pdf:

$$f(x,y) = \begin{cases} 2y\sin(x) & 0 \le x \le \frac{\pi}{2}, \ 0 \le y \le 1\\ 0 & \text{otherwise} \end{cases}$$

Example: Double Integral Calculation

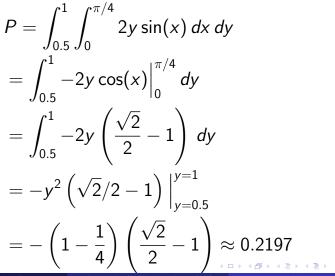
Joint pdf:

$$f(x,y) = \begin{cases} 2y\sin(x) & 0 \le x \le \frac{\pi}{2}, \ 0 \le y \le 1\\ 0 & \text{otherwise} \end{cases}$$

Compute:

$$P(0 \leq X \leq rac{\pi}{4}, \, 0.5 \leq Y \leq 1)$$

Solution: Step-by-Step



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Marginal Probabilities

• Marginalize by integrating out the other variable:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) \, dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) \, dx$$

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• Example:

$$f_X(x) = \int_0^1 2y \sin(x) \, dy = \sin(x)$$
$$f_Y(y) = \int_0^{\pi/2} 2y \sin(x) \, dx = 2y$$

Conditional Probability

• Conditional density (the same as the discrete case):

$$f(x|Y = y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} \\ 0 \end{cases}$$

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• Conditional probability:

$$P(a \le X \le b | Y = y) = \int_a^b f(x | Y = y) \, dx$$

Example: Conditional Density

• Given joint pdf:

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• Marginal:

$$f_Y(y) = 2y$$

• Conditional:

$$f(x|Y = y) = \frac{2y\sin(x)}{2y} = \sin(x)$$

In-Class Exercise: Marginals and Conditionals

Given:

$$f(x,y) = \begin{cases} x^2 + \frac{4}{3}xy + y^2 & (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In-Class Exercise: Marginals and Conditionals

Given:

$$f(x,y) = \begin{cases} x^2 + \frac{4}{3}xy + y^2 & (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Find:

- Marginal densities $f_X(x), f_Y(y)$
- Conditional densities f(x|Y = y), f(y|X = x)

• Just the same as in the discrete case, X and Y are independent if any of the following hold:

$$egin{aligned} &f(x,y)=f_X(x)f_Y(y)\ &f(x|Y=y)=f_X(x)\ &f(y|X=x)=f_Y(y) \end{aligned}$$

In-Class Exercise: Independence

For the two joint densities in previous examples, determine if X and Y are independent.

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Conditional Expectation

• Definition:

$$E[X|Y=y] = \int_{-\infty}^{\infty} xf(x|Y=y) \, dx$$

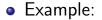
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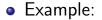
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Compute:

$$E[X|Y=\frac{1}{2}]$$

1

Solution: Conditional Expectation Example

$$E\left[X|Y = \frac{1}{2}\right] = \int_0^1 x \frac{x^2 + \frac{2}{3}x + \frac{1}{4}}{\frac{11}{12}} dx$$
$$= \frac{12}{11} \left(\frac{x^4}{4} + \frac{2}{9}x^3 + \frac{x^2}{8}\right)\Big|_0^1$$
$$= \frac{12}{11} \left(\frac{1}{4} + \frac{2}{9} + \frac{1}{8}\right)$$
$$= \frac{43}{66}$$

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In-Class Exercise: Discrete Conditional Expectation

Given two dice rolls, what is the expected value of the sum given that the first die was a 3?