Joint Probability and Independence for Continuous RV's

Instructor: Shandian Zhe

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- The joint pdf $f(x, y)$ satisfies:
  1. $f(x, y) \geq 0$
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$
  3. $P(a \leq X \leq b,\ c \leq Y \leq d) = \int_c^d \int_a^b f(x, y)\, dx\, dy$

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  2. Evaluate the outside integral:

$$\int_c^d F(y)\, dy$$

Example: Double Integral Calculation

Joint pdf:

$$f(x,y) = \begin{cases} 2y\sin(x) & 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

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Joint pdf:

$$f(x,y) = \begin{cases} 2y\sin(x) & 0 \le x \le \frac{\pi}{2},\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Compute:

$$P(0 \le X \le \frac{\pi}{4},\ 0.5 \le Y \le 1)$$

Solution: Step-by-Step

$$
\begin{aligned}
P &= \int_{0.5}^{1} \int_{0}^{\pi/4} 2y \sin(x)\, dx\, dy \\
&= \int_{0.5}^{1} -2y \cos(x) \Big|_{0}^{\pi/4} dy \\
&= \int_{0.5}^{1} -2y \left(\frac{\sqrt{2}}{2} - 1 \right) dy \\
&= -y^2 \left(\sqrt{2}/2 - 1 \right) \Big|_{y=0.5}^{y=1} \\
&= -\left(1 - \frac{1}{4}\right) \left(\frac{\sqrt{2}}{2} - 1 \right) \approx 0.2197
\end{aligned}
$$

Marginal Probabilities

- Marginalize by integrating out the other variable:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

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- Example:

$$f_X(x) = \int_0^1 2y \sin(x)\, dy = \sin(x)$$

$$f_Y(y) = \int_0^{\pi/2} 2y \sin(x)\, dx = 2y$$

Conditional Probability

- Conditional density (the same as the discrete case):

$$f(x|Y=y) = \begin{cases} \frac{f(x,y)}{f_Y(y)} & f_Y(y) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

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- Conditional probability:

$$P(a \leq X \leq b|Y=y) = \int_a^b f(x|Y=y)\, dx$$

Example: Conditional Density

- Given joint pdf:

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- Marginal:

$$f_Y(y) = 2y$$

- Conditional:

$$f(x|Y = y) = \frac{2y\sin(x)}{2y} = \sin(x)$$

In-Class Exercise: Marginals and Conditionals

Given:

$$f(x,y) = \begin{cases} x^2 + \frac{4}{3}xy + y^2 & (x,y) \in [0,1] \times [0,1] \\ 0 & \text{otherwise} \end{cases}$$

In-Class Exercise: Marginals and Conditionals

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$$f(x, y) = \begin{cases} x^2 + \frac{4}{3}xy + y^2 & (x, y) \in [0, 1] \times [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Find:

- Marginal densities $f_X(x), f_Y(y)$
- Conditional densities $f(x|Y = y), f(y|X = x)$

Independence

- Just the same as in the discrete case, X and Y are independent if any of the following hold:

$$f(x, y) = f_X(x) f_Y(y)$$
$$f(x|Y = y) = f_X(x)$$
$$f(y|X = x) = f_Y(y)$$

In-Class Exercise: Independence

For the two joint densities in previous examples, determine if X and Y are independent.

Conditional Expectation

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Compute:

$$E[X|Y=\frac{1}{2}]$$

Solution: Conditional Expectation Example

$$
\begin{aligned}
E\left[X|Y=\frac{1}{2}\right] &= \int_0^1 x\frac{x^2+\frac{2}{3}x+\frac{1}{4}}{\frac{11}{12}}\,dx \\
&= \frac{12}{11}\left(\frac{x^4}{4}+\frac{2}{9}x^3+\frac{x^2}{8}\right)\Big|_0^1 \\
&= \frac{12}{11}\left(\frac{1}{4}+\frac{2}{9}+\frac{1}{8}\right) \\
&= \frac{43}{66}
\end{aligned}
$$

In-Class Exercise: Discrete Conditional Expectation

Given two dice rolls, what is the expected value of the sum given that the first die was a 3?