Joint Probability and Independence for Discrete RV's

Instructor: Shandian Zhe

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Introduction

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- Collecting multiple variables (e.g., temperature, barometric pressure, wind speed)
- Studying relationships between variables (e.g., smoking and lung cancer)
- Repeated measurements with errors

Joint Probability Density Function

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- Recall: Joint probability of two events A and B is denoted as $P(A \cap B)$.
- Define two events $\{X = a\}$ and $\{Y = b\}$, let us talk about $\{X = a\} \cap \{Y = b\}$: "X is equal to a and Y is equal to b" in English

Joint Probability Density Function

- The joint probability mass function (PMF) is:

$$\begin{aligned} f_{X,Y}(a,b) &= P(X = a, Y = b) \\ &= P(\{X = a\} \cap \{Y = b\}) \end{aligned}$$

- It is also callused joint probability density function.

Joint Probability Density Function

Must satisfy:

$$f_{X,Y}(a,b) \geq 0, \quad \text{for all } a, b$$

$$\sum_i \sum_j f_{X,Y}(a_i, b_j) = 1, \quad \text{where } a_i, b_j \text{ are all possible outcomes for } X \text{ and } Y$$

Example: Binary Communication

- Send a binary message over a wireless network.
- Each bit sent has some probability of being corrupted.

Example: Binary Communication

- S: Sent bit, R: Received bit
- Joint PMF table: $P(S = a, R = b)$:

S \ R	0	1
0	0.45	0.08
1	0.06	0.41

Example: Two Dice Rolls

- X, Y: Outcomes of two dice rolls
- Each outcome (a, b) has probability:

$$f_{X,Y}(a, b) = \frac{1}{36}$$

- Forms a 6×6 probability table.

Marginal Probabilities

- From $f_{X,Y}$, we can recover f_X, f_Y.
- Marginalizing over Y:

$$f_X(a) = P(X = a) = \sum_i f_{X,Y}(a, b_i)$$

- Marginalizing over X:

$$f_Y(b) = P(Y = b) = \sum_i f_{X,Y}(a_i, b)$$

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$$P(R=0)=0.51, \quad P(R=1)=0.49$$
$$P(S=0)=0.53, \quad P(S=1)=0.47$$

In Class Exercise

You flip two fair coins.

- Let H be the total number of heads.
- Let B be the binary number the two coins represent (if heads is a 1 and tails is a 0).

Write down the table for the pdf $f_{H,B}$ and the marginal probabilities f_H, f_B

Conditional Probability

- Definition:

$$f_{X|Y}(a|b) = P(X = a \mid Y = b) = \frac{f_{X,Y}(a,b)}{f_Y(b)}$$

Example: Binary Communication

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Example: Binary Communication

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- What is $f_{R|S}(1|1)$ and $f_{R|S}(0|0)$?

In Class Exercise

You flip two fair coins.

- Let H be the total number of heads.
- Let B be the binary number the two coins represent (if heads is a 1 and tails is a 0).

Compute $f_{H|B}$ and $f_{B|H}$

Independence

- Recall: two events A and B are independent:

$$
\begin{aligned}
P(A, B) &= P(A)P(B), \quad \text{or} \\
P(A|B) &= P(A), \quad \text{or} \\
P(B|A) &= P(B)
\end{aligned}
$$

Independence

- Two random variables X, Y are independent if:

$$f_{X,Y}(a,b) = f_X(a)f_Y(b) \quad \text{for all } a, b$$

- equivalent to our first definition of independence for events, with the added *rule*: it must hold for <u>all</u> possible outcomes for X and Y. In other words, X and Y are independent of each other if all the events defined by X are independent from all the events defined by Y.

Independence: Insight

- The independence definition above says only that events of the form $\{X = a\}$ and $\{Y = b\}$ are independent

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- The independence definition above says only that events of the form $\{X = a\}$ and $\{Y = b\}$ are independent
- However, this implies that any two events $A = \{a_i\}$ and $B = \{b_j\}$ defined using X and Y are also independent. Why?

In-Class Exercise

S \ R	0	1
0	0.45	0.08
1	0.06	0.41

- Are R and S independent?

In-Class Exercise

Let X and Y be two dice rolls. Verify that $X \in \{1, 3, 4\}$ is independent of $Y \in \{1, 2\}$.