Expectation and Variance

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Expectation

Intuitively, it means on **average**, what the value does a random variable take?

Expectation: Discrete Random Variables

The **expectation of a discrete random variable** X taking values $\{a_i\}$ with probability mass function p is given by:

$$\mathbb{E}[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i).$$

The expectation is the value that you would expect on average if you repeat an experiment many times.

Example: Bernoulli Expectation

What is the expectation of $X \sim Ber(p)$?

$$\mathbb{E}[X] = \sum_{k=0}^{1} kp(k) = 0 \cdot (1 - p) + 1 \cdot p = p$$

Example: Geometric Expectation

What is the expectation of $X \sim Geo(p)$?

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{1}{p}$$

In-Class Exercise

What is the expectation of a six-sided die roll?

$$\mathbb{E}[X] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$$

Expectation: Continuous Random Variables

The **expectation of a continuous random variable** X with probability density function f is given by:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Example: Exponential Distribution

Example: What is the expectation of $X \sim Exp(\lambda)$?

$$\mathbb{E}[X] = \int_0^{\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}$$

Example: Gaussian Distribution

Example: What is the expectation of $X \sim \mathcal{N}(\mu, \sigma^2)$? It is μ!

Linearity of Expectation

If X and Y are random variables and $a, b \in \mathbb{R}$, then:

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$$

Example: Sum of Dice Rolls

If we roll 10 dice and sum them up, what is the expected value of the result?

$$\mathbb{E}[S] = \mathbb{E}[10 \cdot X] = 10 \cdot \mathbb{E}[X] = 35.$$

In-Class Exercise: Binomial Distribution

Remember that if $X \sim Bin(n, p)$, then X is the sum of n Bernoulli random variables, $X_i \sim Ber(p)$. Use the linearity of expectation to compute $\mathbb{E}[X]$.

$$
\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\
&= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\
&= np
\end{aligned}
$$

Expectation of a Function

Expectation of a function $g : \mathbb{R} \to \mathbb{R}$ of a random variable:

Discrete case:

$$\mathbb{E}[g(X)] = \sum_i g(a_i)\, p(a_i)$$

Continuous case:

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Variance

The **variance** of a random variable X is given by:

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

The variance describes how *spread out* a random variable's distribution is.

Standard Deviation

The **standard deviation**, defined as the square root of the variance,

$$\text{Std}(X) = \sqrt{\text{Var}(X)},$$

is often a more useful description of the spread (it's in the the same units as $\mathbb{E}[X]$)

Example: Bernoulli Variance

Example: The variance of Bernoulli random variable, $X \sim Ber(p)$:

$$\text{Var}(X) = p(1-p) \quad (\textit{Why?})$$

Example: Gaussian Distribution

Example: What is the variance of $X \sim \mathcal{N}(\mu, \sigma^2)$? It is σ^2

Equivalent Variance Formula

An equivalent formula for variance is:

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \quad (\textit{Why})$$

Variance Scaling Property

Variance after a scaling and shift, $a, b \in \mathbb{R}$:

$$\text{Var}(aX + b) = a^2\text{Var}(X) \quad (\textit{Why})$$

Example: Die Roll Variance

What is the variance of a six-sided die roll?

$$\mathrm{Var}(X) = \sum_{k=1}^{6} \frac{1}{6} k^2 - \mathbb{E}[X]^2$$
$$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) - \left(\frac{7}{2}\right)^2$$
$$= \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92$$

Standard deviation:

$$\sqrt{\mathrm{Var}(X)} \approx 1.71$$