Continuous Random Variables

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Random Variables

A random variable on a sample space Ω is just a function:

$$X : \Omega \to \mathbb{R}$$

So far, our sample spaces have all been discrete sets, and thus the output of our random variables have been restricted to discrete values. What if the sample space is continuous, such as $\Omega = \mathbb{R}$?

Random Variables

A random variable on a sample space Ω is just a function:

$$X : \Omega \to \mathbb{R}$$

If Ω is continuous, X can take on a continuum of values.

Example: Continuous Random Variable

- Record time elapsed from start of class until the last person arrives.
- T takes values from 0 to 80 minutes.
- What is the probability $P(T = 5)$?
- As measurement precision increases, $P(T = 5)$ approaches 0. Why?
- However, $P(5 \leq T \leq 6)$ is nonzero.

Probability Density Function (PDF)

- A PDF $f(x)$ defines probabilities via integration:

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

- Properties:
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Cumulative Distribution Function (CDF)

- The CDF is defined as:

$$F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)\, dx.$$

Uniform Distribution

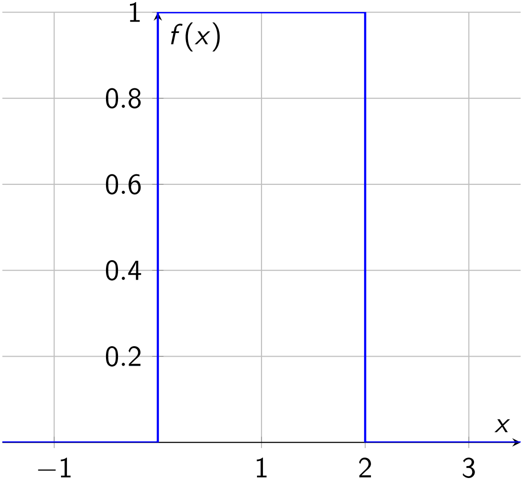
- PDF:

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & x \in [\alpha, \beta] \\ 0 & \text{otherwise} \end{cases}$$

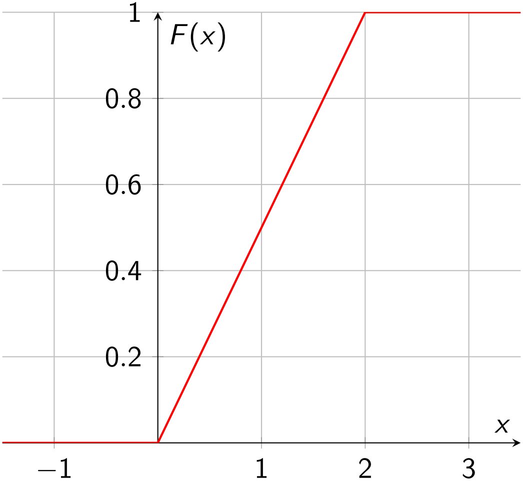
- Notation: $X \sim U(\alpha, \beta)$
- CDF:

$$F(a) = \frac{a-\alpha}{\beta-\alpha}, \quad \alpha \leq a \leq \beta$$

Uniform Distribution (PDF)



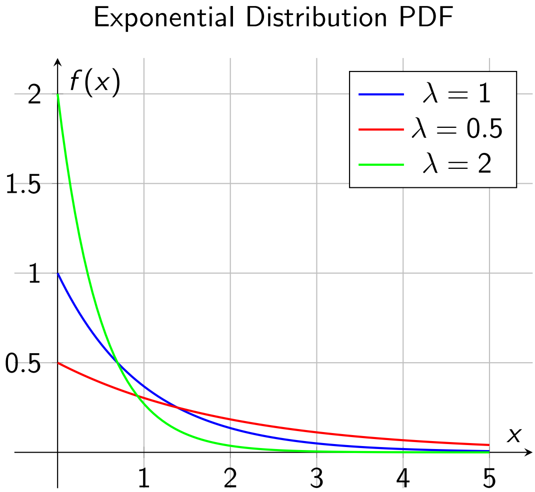
Uniform Distribution (CDF)



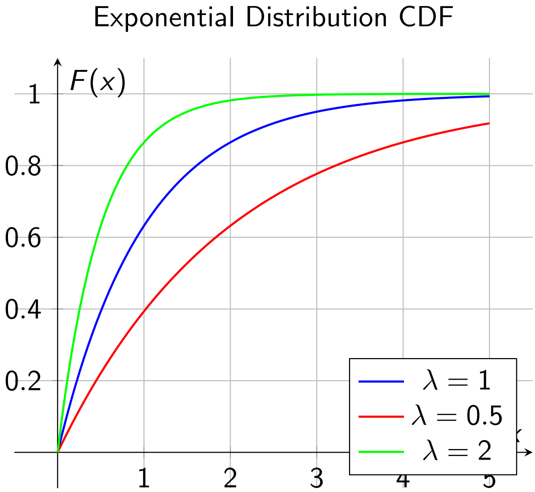
Exponential Distribution

- PDF: $f(x) = \lambda e^{-\lambda x}$
- CDF: $F(a) = 1 - e^{-\lambda a}$
- Notation: $X \sim Exp(\lambda)$
- Models waiting times between Poisson-distributed events.

Exponential Distribution (PDF)



Exponential Distribution (CDF)



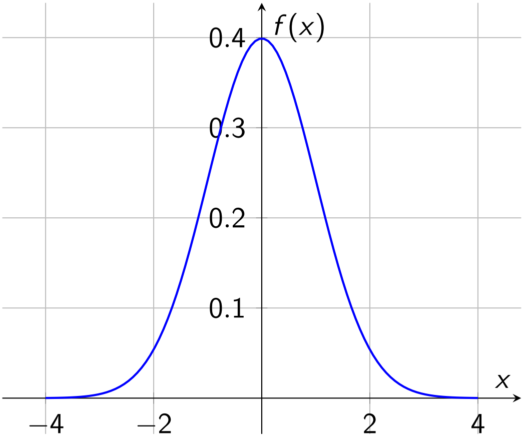
Normal (Gaussian) Distribution

- PDF:

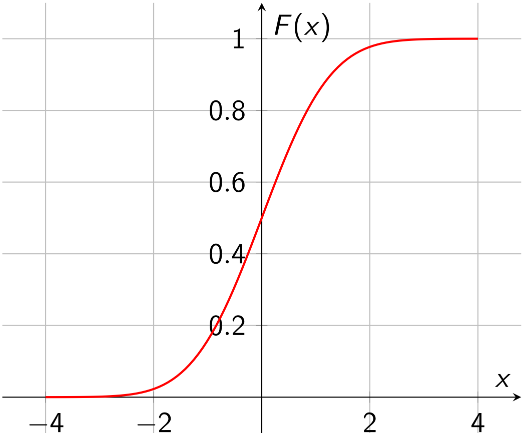
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

- Notation: $X \sim N(\mu, \sigma^2)$
- No closed-form CDF, computed numerically.
- Models many natural phenomena (e.g., measurement noise).

Gaussian Distribution (PDF)



Gaussian Distribution (CDF)



Pareto Distribution

- PDF:

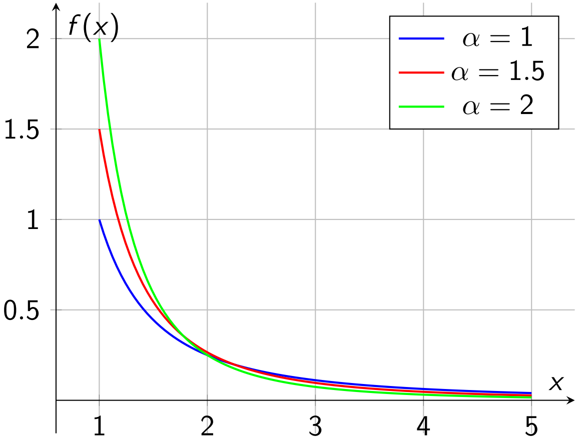
$$f(x) = \frac{\alpha}{x^{\alpha+1}}, \quad x \geq 1$$

- CDF:

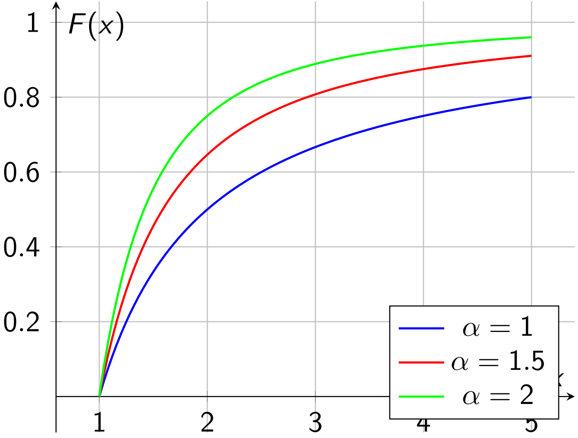
$$F(a) = 1 - \frac{1}{a^{\alpha}}, \quad a \geq 1$$

- Models wealth distribution, file sizes, etc.
- The richest few own a disproportionate amount of total wealth.
- A few big corporations dominate the market.
- A small number of files in Internet are huge, while most are small.

Pareto Distribution (PDF)



Pareto Distribution (CDF)



Quantiles

- The pth quantile q_p satisfies:

$$F(q_p) = P(X \leq q_p) = p.$$

- The 50th quantile is the **median**.