Independence

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Independence

An event A is independent of an event B when:

$$P(A \mid B) = P(A)$$

In English: "the probability of A does not depend on whether B happens." If A and B are not independent, we say they are dependent.

Breaking Down the Equation

Using the definition of conditional probability:

$$
\begin{aligned}
& P(A|B) = P(A) \\
\Leftrightarrow \quad & \frac{P(A \cap B)}{P(B)} = P(A) \\
\Leftrightarrow \quad & \underline{P(A \cap B) = P(A)P(B)}
\end{aligned}
$$

- This is an equivalent definition of independence.

Continuing on

Definition of A and B independent

$$\begin{aligned} & P(A \cap B) = P(A)P(B) \\ \Leftrightarrow \quad & \frac{P(A \cap B)}{P(A)} = P(B) \\ \Leftrightarrow \quad & \underline{P(B|A) = P(B)} \end{aligned}$$

- $P(B|A) = P(B)$ is another definition of A and B independent.
- This tells us independence is a *symmetric* property: $P(A|B) = P(A)$ is equivalent to $P(B|A) = P(B)$

In-Class Problem

A fair die is thrown twice.

- A: The sum of the values is 5.
- B: At least one throw is a 2.

Questions

1. Calculate $P(A \mid B)$.
2. Are events A and B independent?

In-Class Problem

Two urns:

- Urn 1: 4 black balls, 3 white balls.
- Urn 2: 2 black balls, 2 white balls.

Pick one urn at random, then select a ball from the urn.

Questions

1. Is the event "picking Urn 1" independent of the event "picking a white ball"?
2. What if Urn 2 had 8 black balls and 6 white balls?

In-Class Problem

A system has a main power supply and an auxiliary power supply:

- Main supply failing: 10% chance.
- Auxiliary supply failing: 10% if main supply works, 15% if main supply fails.

Question

Is the auxiliary supply failing independent of main supply failing?

Intuitively

- From our English translation of independence:
- If A and B are independent, the probability of A would be the same if B happens or if B does *not* happen, that is, if B^c happens. Let's check

Intuitively

$$P(A \cap B) = P(A)P(B)$$

(Definition of independence)

Intuitively

$$P(A \cap B) = P(A)P(B)$$
$$\Leftrightarrow P(A - B^c) = P(A)P(B)$$

(Definition of set minus)

Intuitively

$$P(A - B^c) = P(A)P(B)$$

$$\Leftrightarrow P(A) - P(A \cap B^c) = P(A)P(B)$$

(Difference rule)

Intuitively

$$P(A) - P(A \cap B^c) = P(A)P(B)$$
$$\Leftrightarrow P(A) - P(A \cap B^c) = P(A)(1 - P(B^c))$$

(Complement rule)

Intuitively

$$P(A) - P(A \cap B^c) = P(A)(1 - P(B^c))$$
$$\Leftrightarrow \underline{P(A \cap B^c) = P(A)P(B^c)}$$

(Subtract $P(A)$ from both sides and multiply by -1)

Intuitively

$$P(A) - P(A \cap B^c) = P(A)(1 - P(B^c))$$
$$\Leftrightarrow \underline{P(A \cap B^c) = P(A)P(B^c)}$$

(Subtract $P(A)$ from both sides and multiply by -1)

- This gives the definition that A and B^c are independent

Summary: Definitions of Independence

The events A and B are independent if any of the following conditions hold:

1. $P(A \mid B) = P(A)$
2. $P(B \mid A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$
4. Replace B with B^c:

$$P(A \mid B^c) = P(A) \quad \text{or} \quad P(B^c \mid A) = P(B^c)$$
$$\text{or} \quad P(A \cap B^c) = P(A)P(B^c)$$