# Conditional Probability

## CS 3130/ECE 3530: Probability and Statistics for Engineers

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# Review of Event Translations

**English Translation for Events:**

- $A \cap B$: "Both events $A$ and $B$ happen"
- $A \cup B$: "Either event $A$ or $B$ (or both) happens"
- $A^c$: "Event $A$ does not happen"

# Set Theory Rules

Try drawing Venn diagram of these

- Definition of Set Difference:

$$A - B = A \cap B^c \quad \text{“}A \text{ happens, but } B \text{ does not”}$$

- Commutative Law:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

- Associative Law:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

# Set Theory Rules

- Distributive Law:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

- DeMorgan's Laws:

$$(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$$

# Probability

## Definition

A **probability function** on a finite sample space $\Omega$ assigns every event $A \subseteq \Omega$ a number in $[0, 1]$, such that

1. $P(\Omega) = 1$
2. $P(A \cup B) = P(A) + P(B)$ when $A \cap B = \emptyset$

$P(A)$ is the **probability** that event $A$ occurs.

# Equally Likely Outcomes

The number of elements in a set $A$ is denoted $|A|$.

If $\Omega$ has a finite number of elements, and each is equally likely, then the probability function is given by

$$P(A) = \frac{|A|}{|\Omega|}$$

Example: Rolling a 6-sided die

- $P(\{1\}) = 1/6$
- $P(\{1, 2, 3\}) = 1/2$

# Exercise

You are picking a number out of a hat, which contains the numbers 1 through 100. What are the following events and their probabilities?

- The number has a single digit
- The number has two digits
- The number is a multiple of 4
- The number is not a multiple of 4
- The sum of the number's digits is 5

# Permutations

A **permutation** is an ordering of an $n$-tuple. For instance, the $n$-tuple $(1, 2, 3)$ has the following permutations:

$$(1, 2, 3), (1, 3, 2), (2, 1, 3)$$
$$(2, 3, 1), (3, 1, 2), (3, 2, 1)$$

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$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2$$

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How many ways can you rearrange $(1, 2, 3, 4)$?

# Probability Rules

- **Inclusion-Exclusion Rule:**

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- **Complement Rule:**

$$P(A^c) = 1 - P(A)$$

- **Difference Rule:**

$$P(A - B) = P(A) - P(A \cap B)$$

# Probability Rules

Exercise: Try deriving these rules from the definition of a probability function. Draw a Venn diagram to convince yourself they work.

# Conditional Probability

$P(A \mid B) =$ "The probability of event $A$ given $B$ happened"

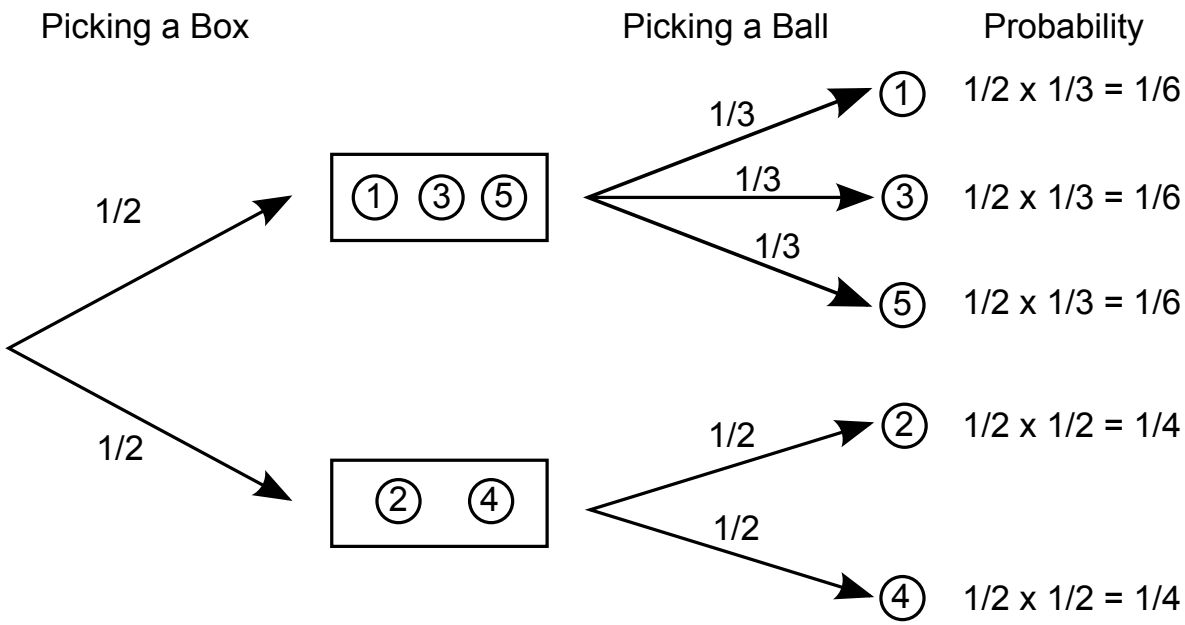
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

# Multiplication Rule

$$P(A \cap B) = P(A \mid B)P(B)$$

# Tree Diagrams Example

You are given two boxes with balls numbered 1 - 5. One box contains balls 1, 3, 5, and the other contains balls 2 and 4. You pick a box at random, then a ball from that box. What is the probability of picking a 2?



# Sampling With Replacement

**Problem:**

I have a box with 10 red and 10 green balls. I draw 2 *with* replacement. What is $P(\text{2 red balls})$?

$$P(R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1) = \frac{10}{20} \cdot \frac{10}{20} = \frac{1}{4} = 0.25$$

# Sampling Without Replacement

**Problem:**

I have a box with 10 red and 10 green balls. I draw 2 *without* replacement. What is $P(\text{2 red balls})$?

$$P(R_1 \cap R_2) = P(R_1)P(R_2 \mid R_1) = \frac{10}{20} \cdot \frac{9}{19} = \frac{9}{38} \approx 0.24$$

# In-Class Problem

A fair die is thrown twice. Let $A$ be the event that the sum of values is 5, and $B$ the event that at least one throw is a 2. Calculate $P(A \mid B)$.

# In-Class Problem: Urn Example

You have two urns:

- Urn 1: 4 black balls and 3 white balls.
- Urn 2: 2 black balls and 2 white balls.

You pick an urn at random and then select a ball. What is the probability that the ball is white?