Graph Cuts

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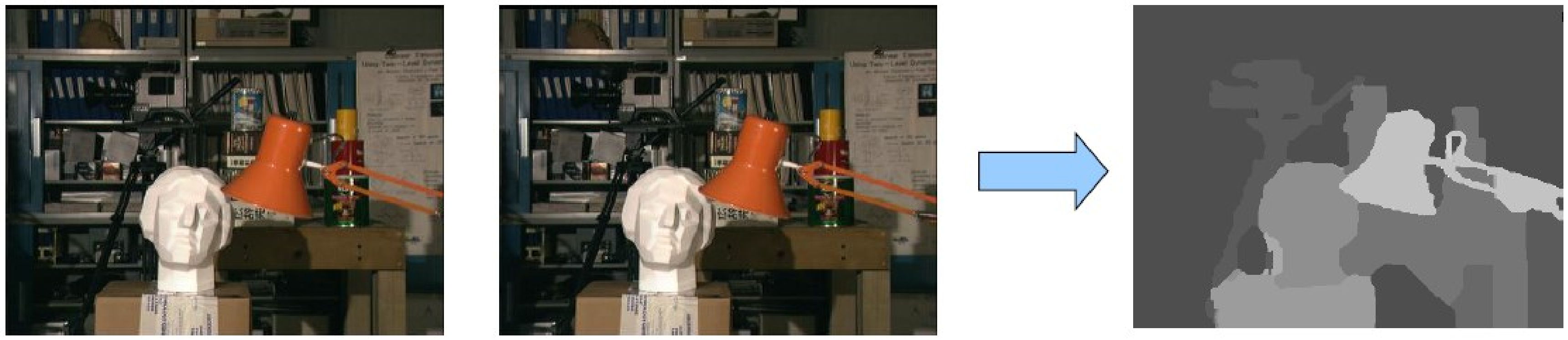
Outline

- Introduction
- Pseudo-Boolean Functions
- Submodularity
- Max-flow / Min-cut Algorithm
- Alpha-Expansion

Segmentation Problem

[Boykov and Jolly'2001,
Rother et al. 2004]

Stereo Reconstruction



Left Camera Image — Right Camera Image — Dense Stereo Result

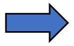
- Choose the disparities from the discrete set: $(1,2,\ldots,L)$

Image Denoising

Original

Denoised image

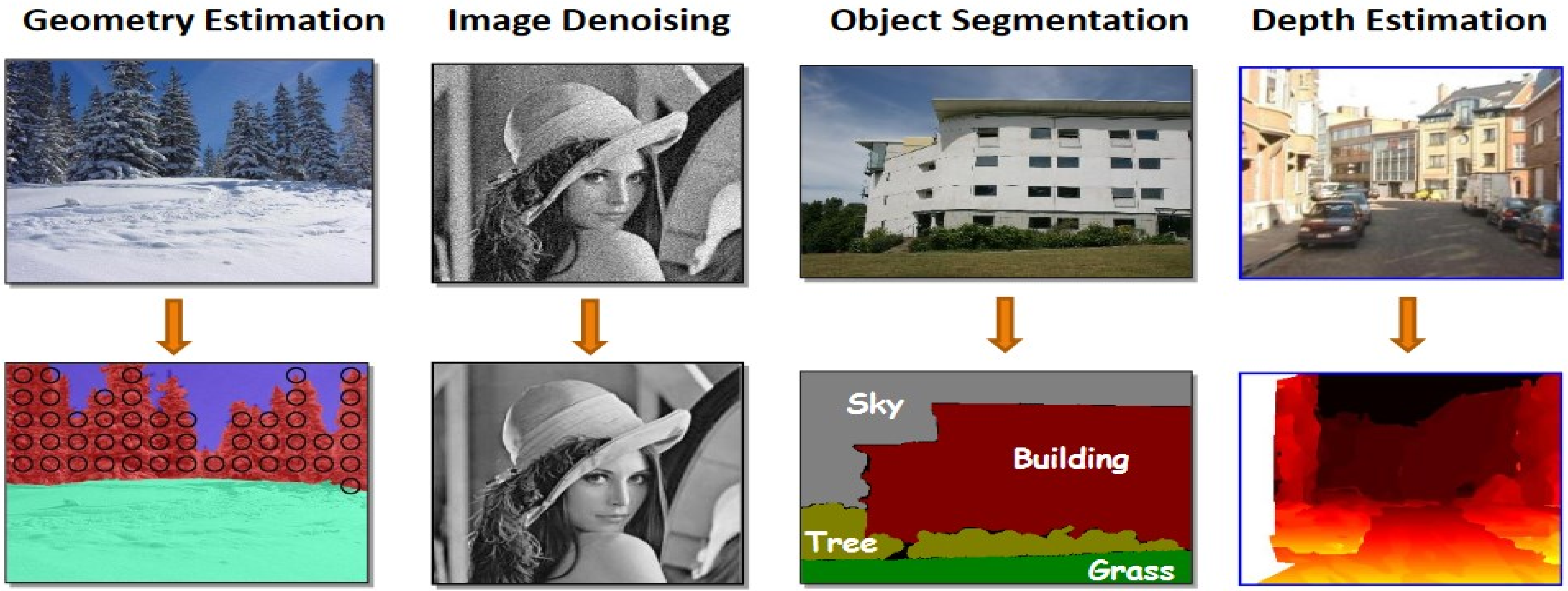
Semantic Labeling (Building, ground, sky)



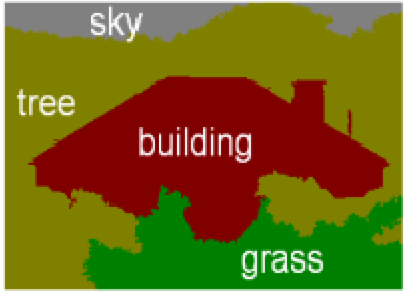
[Hoiem, Efros, Hebert, *IJCV*, 2007]

Image Labeling Problems

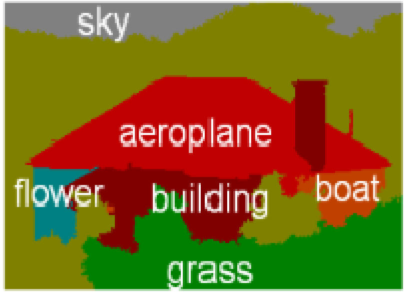
Assign a label to each image pixel



Labeling is highly structured



Possible labeling

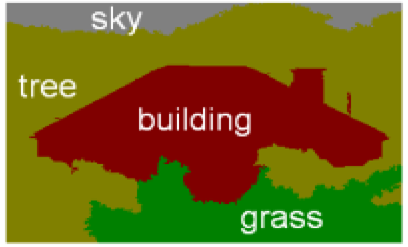


Impossible labeling

Image Courtesy: Lubor Ladicky

Labeling is highly structured

- Labelings highly structured
- Labels highly correlated with very complex dependencies



- Neighbouring pixels tend to take the same label
- Low number of connected components
- Classes present may be seen in one image
- Geometric / Location consistency
- Planarity in depth estimation
- … many others (task dependent)

Slide Courtesy: Lubor Ladicky

Image Labeling Problems

- Labelings highly structured
- Labels highly correlated with very complex dependencies
- Independent label estimation too hard
- Whole labelling should be formulated as one optimisation problem
- Number of pixels up to millions
  - Hard to train complex dependencies
  - Optimisation problem is hard to infer

Slide Courtesy: Lubor Ladicky

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Pseudo Boolean Functions (PBF)

- Variables: $x_1, x_2, \ldots, x_n \in \{0,1\}$
- Negations: $\bar{x}_i = 1 - x_i \in \{0,1\}$
- Pseudo-Boolean Functions (PBF): $f : \{0,1\}^n \rightarrow R$
  - » Maps a Boolean vector to a real number.
- Has unique multi-linear representation:
  - » For example:

$$f(x_1, x_2, x_3, x_4) = 2 - 3x_2x_4 + 5x_1x_2x_3$$

[Boros&Hammer'2002]

Posiforms for Pseudo-Boolean functions (PBF)

- Posiforms: Non-negative multi-linear polynomial except maybe the constant terms.

$$f(x_1, x_2, x_3, x_4) = 2 - 3x_2x_4 + 5x_1x_2x_3$$
$$= 2 - 3(1 - \overline{x_2})x_4 + 5x_1x_2x_3$$
$$= 2 - 3x_4 + 3\overline{x_2}x_4 + 5x_1x_2x_3$$
$$= 2 - 3(1 - \overline{x_4}) + 3\overline{x_2}x_4 + 5x_1x_2x_3$$
$$\phi = -1 + 3\overline{x_4} + 3\overline{x_2}x_4 + 5x_1x_2x_3$$

- Several posiforms exist for a given function.
- Provides bounds for minimization, e.g. $\phi = -1$

[Boros&Hammer'2002]

Set Functions are Pseudo Boolean Functions (PBF)

- Finite ground set $V = \{1,2,...,n\}$

- Set function (Input - subset of V, output - real number)

$$f_s : 2^V \to R$$

- 1-1 correspondence exists between $x_1, x_2, ..., x_n \in \{0,1\}$ and subset S of V.

$$V = \{1,2,3,4\}$$

$$\{x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 1\} \Leftrightarrow (1,2,4)$$

$$x_i = 1 \Longleftrightarrow i \in S$$

$$x_i = 0 \Longleftrightarrow i \notin S$$

Set Functions are Pseudo Boolean Functions (PBF)

- Consider a PBF $f(x_1, x_2, x_3, x_4) = 2 - 3x_2x_4 + 5x_2x_3$
- Equivalent to a set function

$$f_s(\{1,2\}) = 2 - 3(1)(0) + 5(1)(0) = 2$$

$$f_s(\{2,3\}) = 2 - 3(1)(0) + 5(1)(1) = 7$$

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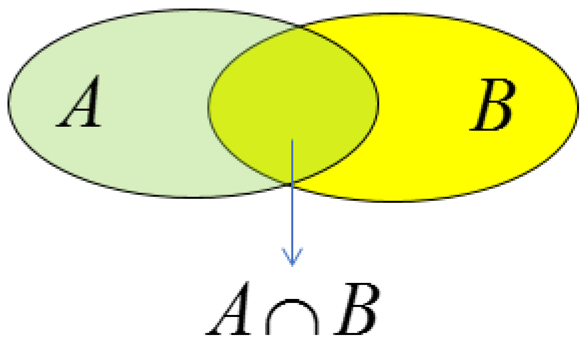
Submodular set functions (Union-Intersection)



$A, B \subseteq V$

- A set function $f : 2^V \to R$ is submodular if and only if:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \forall A, B \subseteq V$$

Equivalent Definitions

- Diminishing gains: for all $A \subseteq B$

$$F(A \cup s) - F(A) \geq F(B \cup s) - F(B)$$

Slide Courtesy: Krause, Jegelka

Questions

How do I prove my problem is submodular?

Why is submodularity useful?

Slide Courtesy: Krause, Jegelka

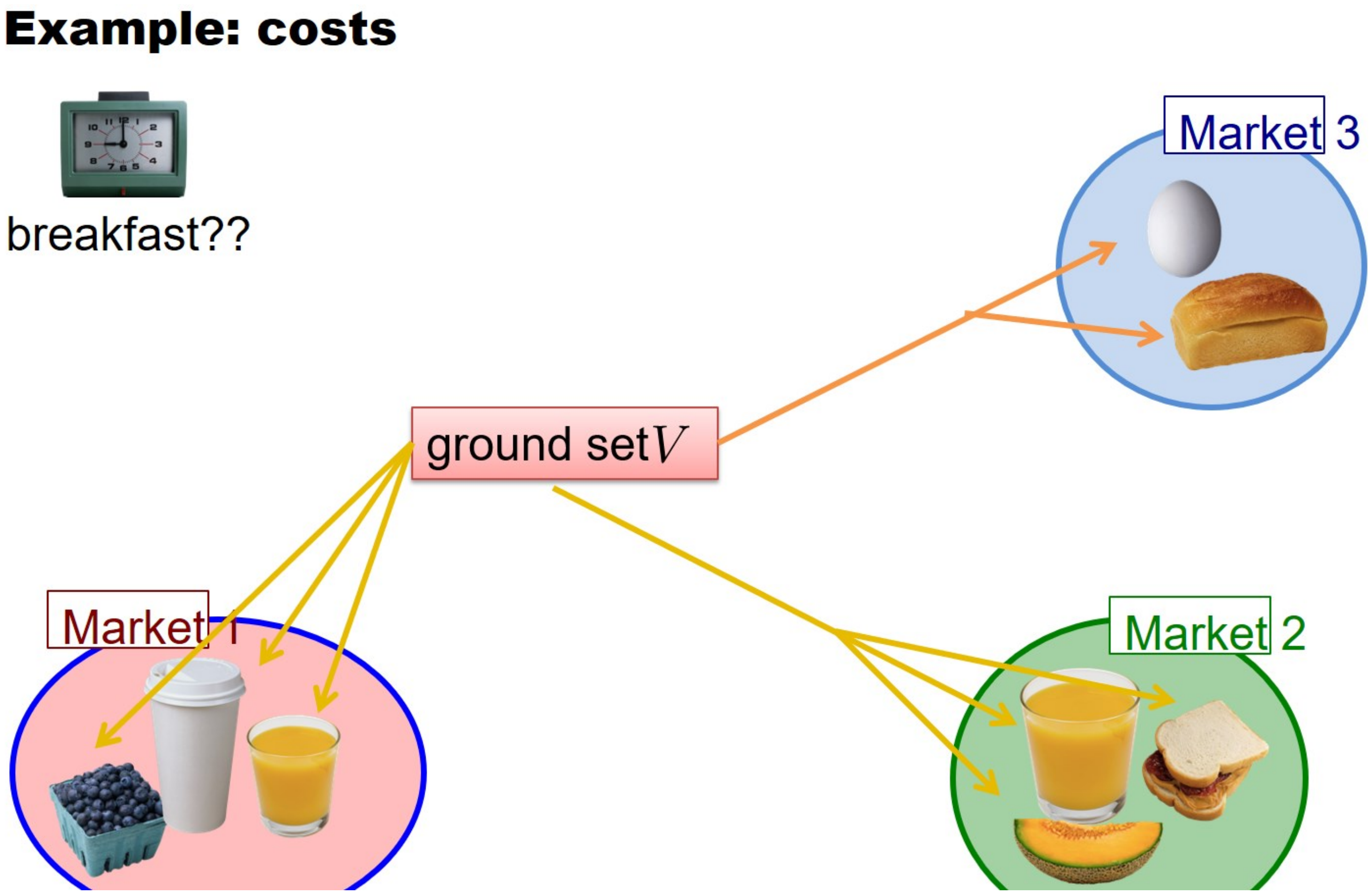
Submodularity Example

Example: costs

breakfast??

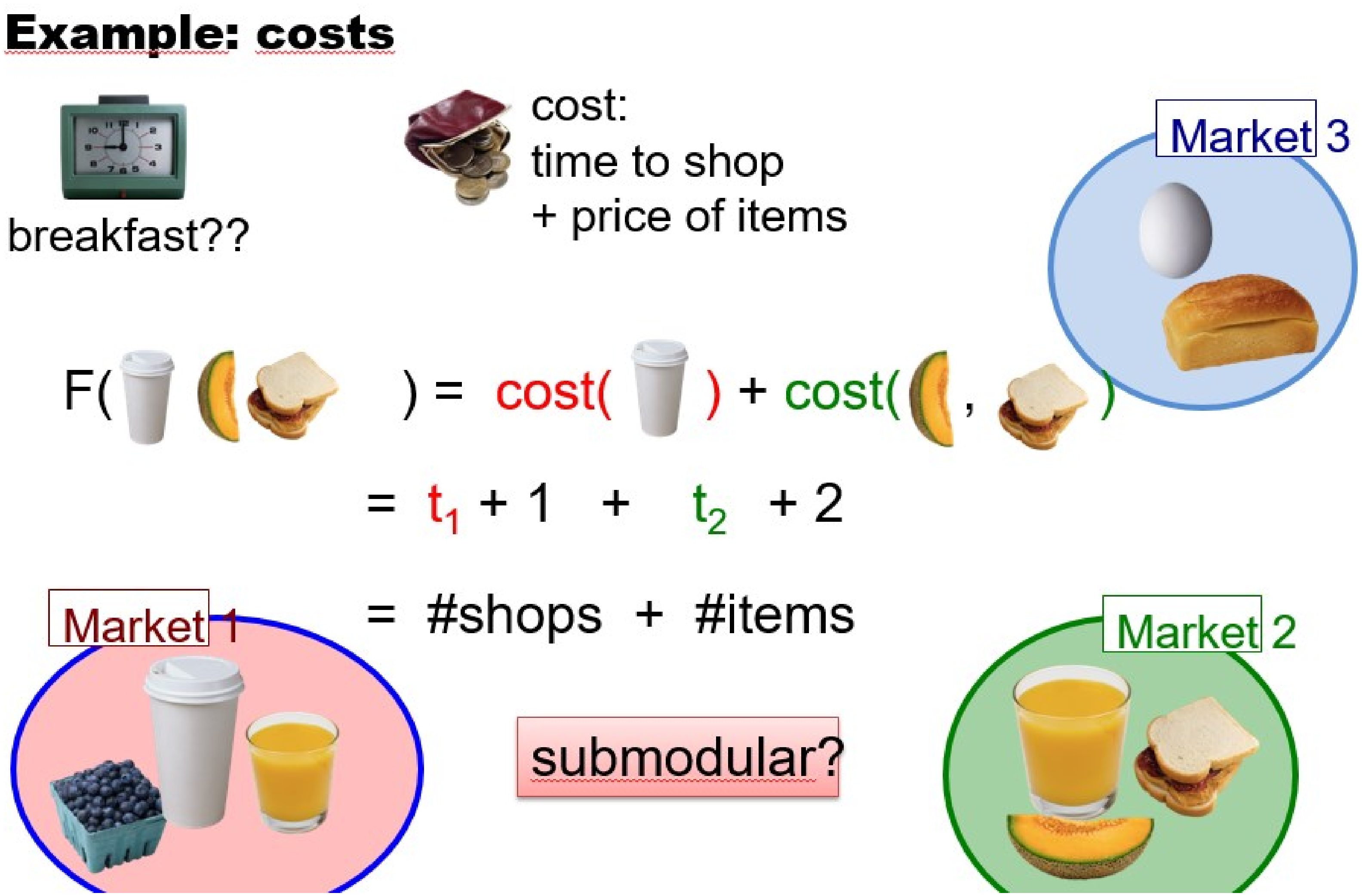
Slide Courtesy: Krause, Jegelka

Submodularity Example



Slide Courtesy: Krause, Jegelka

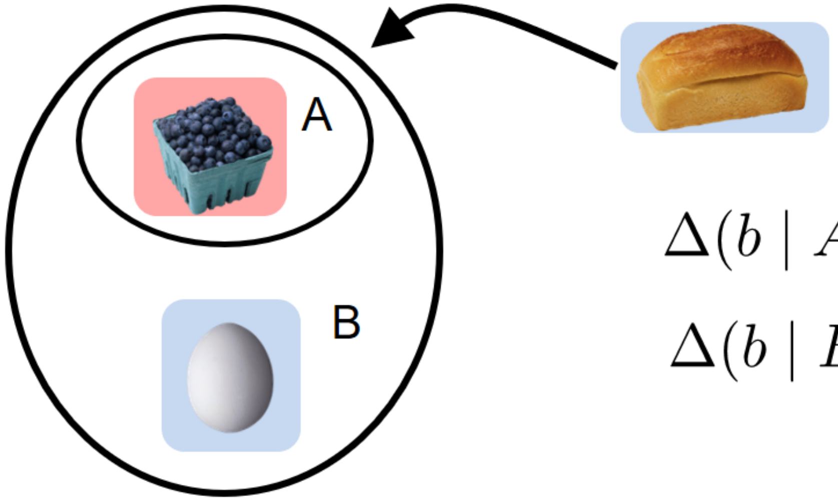
Submodularity Example



Slide Courtesy: Krause, Jegelka

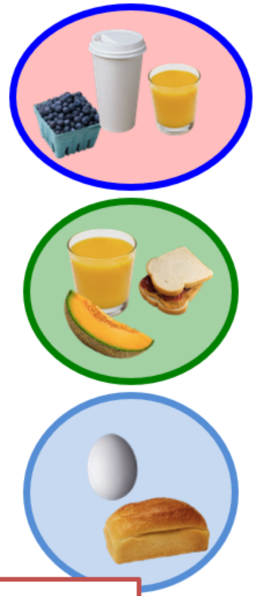
Submodularity Example

Shared fixed costs



$$\Delta(b \mid A) = 1 + t_3$$

$$\Delta(b \mid B) = 1$$



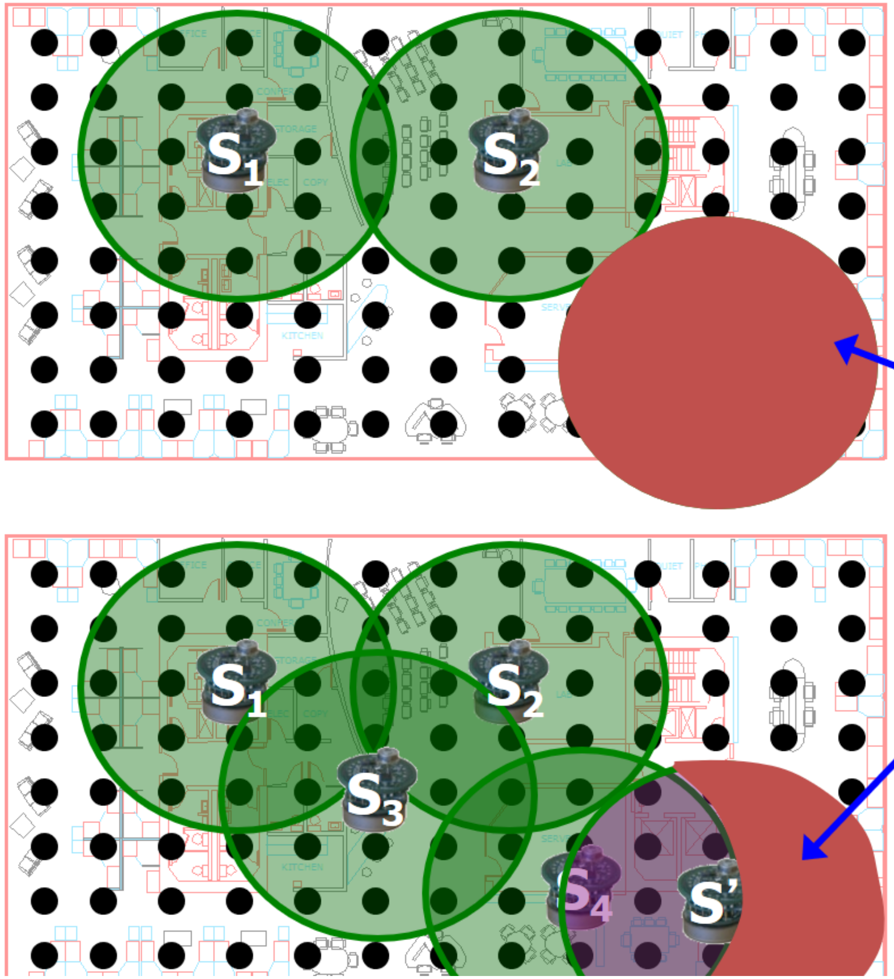
marginal cost: #new shops + #new items

decreasing ➔ cost is submodular!

- shops: shared fixed cost
- economies of scale

Slide Courtesy: Krause, Jegelka

Set cover is submodular



$A=\{s_1,s_2\}$

$F(A \cup \{s'\}) - F(A)$

$\geq$

$F(B \cup \{s'\}) - F(B)$

$B = \{s_1,s_2,s_3,s_4\}$

Slide Courtesy: Krause, Jegelka

Submodular set functions (Union-Intersection)

$$f(A)+f(B)\geq f(A\cup B)+f(A\cap B),\forall A,B\subseteq V$$

Let us consider a very simple case with only two variables x_1 and x_2.

$$V=\{1,2\},A=\{1\},B=\{2\}$$

$f(0,0)$	$f(0,1)$
$f(1,0)$	$f(1,1)$

Using submodularity, we have:

$$f(x_1=1,x_2=0)+f(x_1=0,x_2=1)\geq$$
$$f(x_1=1,x_2=1)+f(x_1=0,x_2=0)$$
$$f(1,0)+f(0,1)\geq f(1,1)+f(0,0)$$

- Main diagonal elements are smaller than off-diagonal ones.
- Blue is larger than red.

Quadratic Pseudo Boolean Functions (QPBF)

- Example of quadratic pseudo Boolean functions

$$f(x_1, x_2, x_3, x_4) = 1 + x_1 - 3x_2 + x_1x_2 + 5x_3x_4$$

[Boros&Hammer'2002]

Submodular Quadratic Pseudo Boolean Functions

- A QPBF is submodular if and only if all quadratic coefficients are non-positive.

$$f_3(x_1, x_2, x_3) = 15 + x_1 - 3x_2 - \underline{x_1 x_2} - \underline{5x_2 x_3}$$

Example for submodular QPBF

$$f_3(x_1, x_2, x_3) = 15 + x_1 - 3x_2 - 3x_1x_3 - 5x_2x_3$$

$$V = \{1,2,3\}, A = \{1,2\}, B = \{2,3\}$$

$$A \cup B = \{1,2,3\}, A \cap B = \{2\}$$

$$f(A) = 15 + 1 - 3(1) - 3(1)(0) - 5(1)(0) = 13$$

$$f(B) = 15 + 0 - 3(1) - 3(0)(1) - 5(1)(1) = 7$$

$$f(A \cup B) = 15 + 1 - 3(1) - 3(1)(1) - 5(1)(1) = 5$$

$$f(A \cap B) = 15 + 0 - 3(1) - 3(0)(0) - 5(1)(0) = 12$$

$$\Rightarrow f(A) + f(B) \geq f(A \cup B) + f(A \cap B), (13 + 7 > 5 + 12)$$

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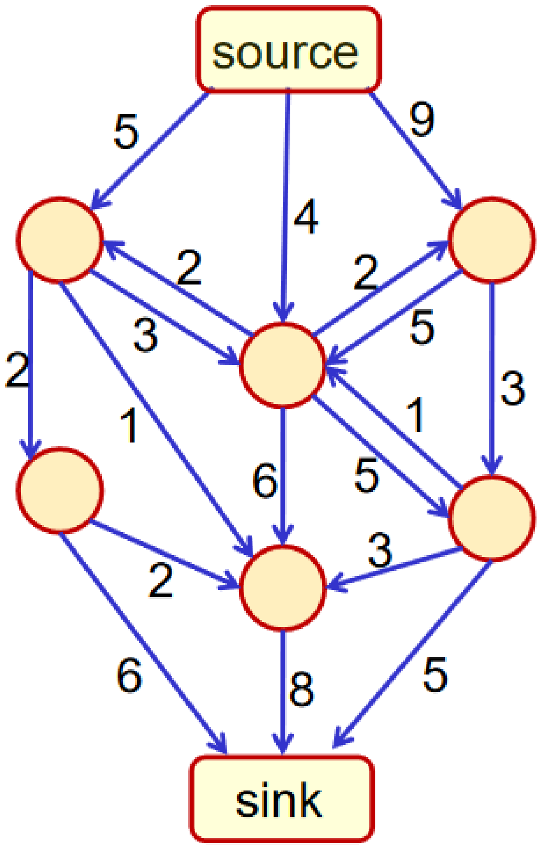
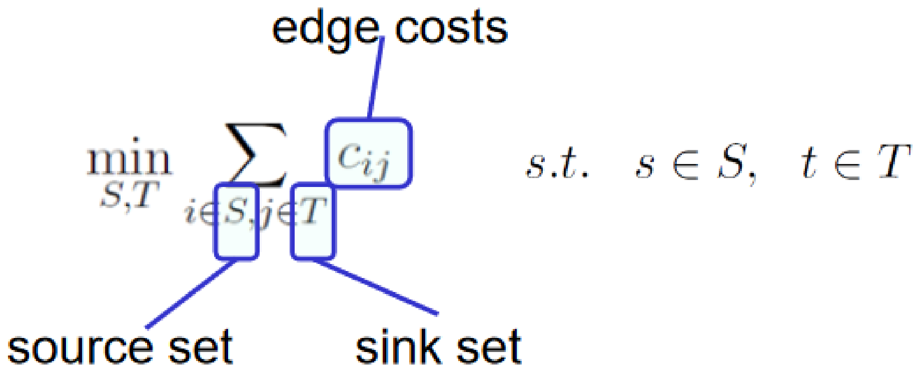
Max-flow/Min-cut

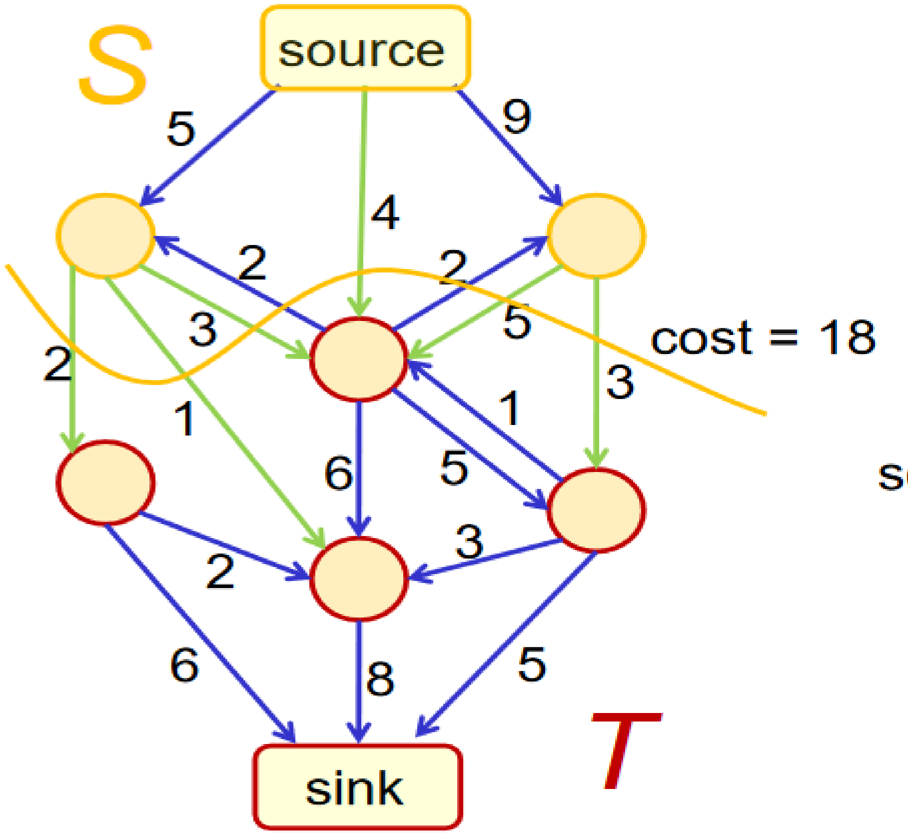




$$\min_{S,T} \sum_{i \in S, j \in T} c_{ij} \qquad s.t. \quad s \in S, \quad t \in T$$

Image courtesy: Lubor Ladicky

Max-flow/Min-cut



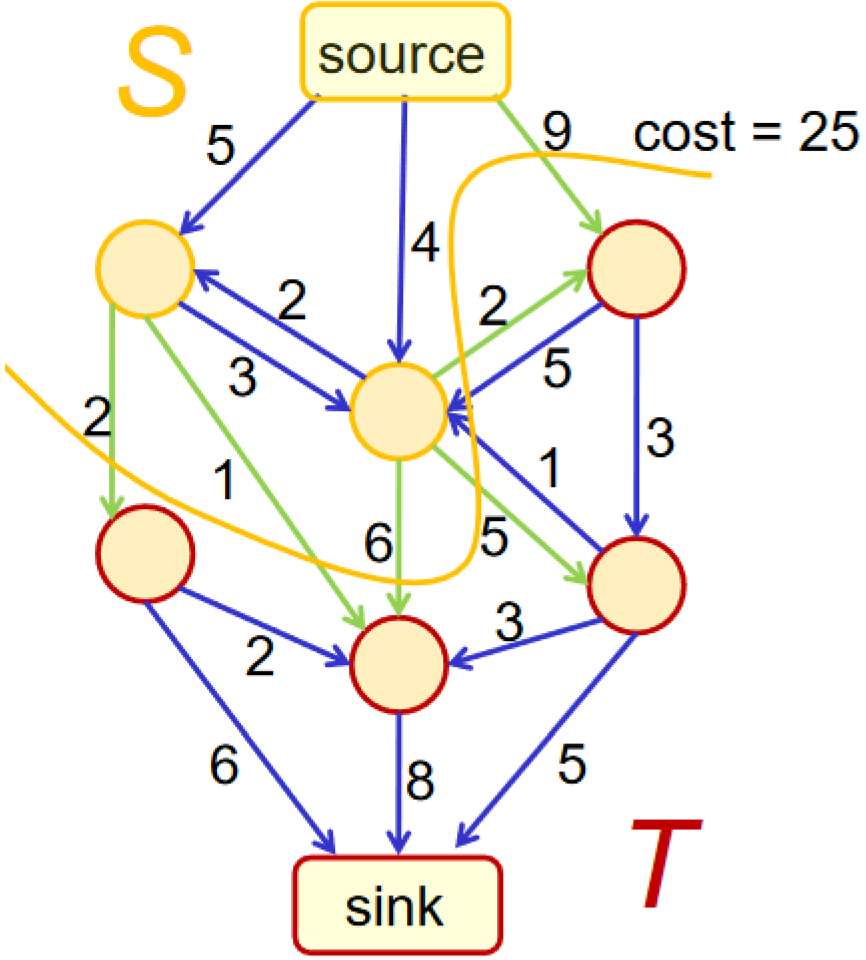
edge costs

$$\min_{S,T} \sum_{i \in S, j \in T} c_{ij} \qquad s.t. \quad s \in S, \quad t \in T$$

source set

sink set

Max-flow/Min-cut



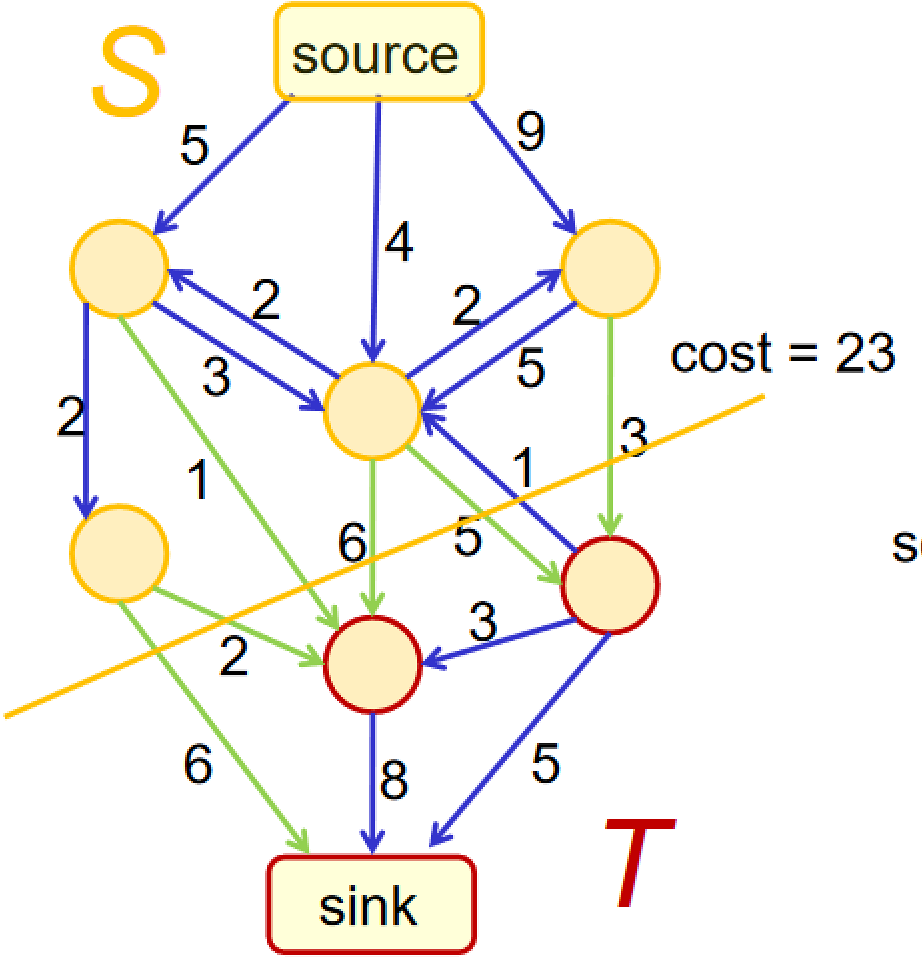
edge costs

$$\min_{S,T} \sum_{i \in S, j \in T} c_{ij} \qquad s.t. \quad s \in S, \quad t \in T$$

source set

sink set

Max-flow/Min-cut



$$\min_{S,T} \sum_{i \in S, j \in T} c_{ij} \qquad s.t. \quad s \in S, \quad t \in T$$

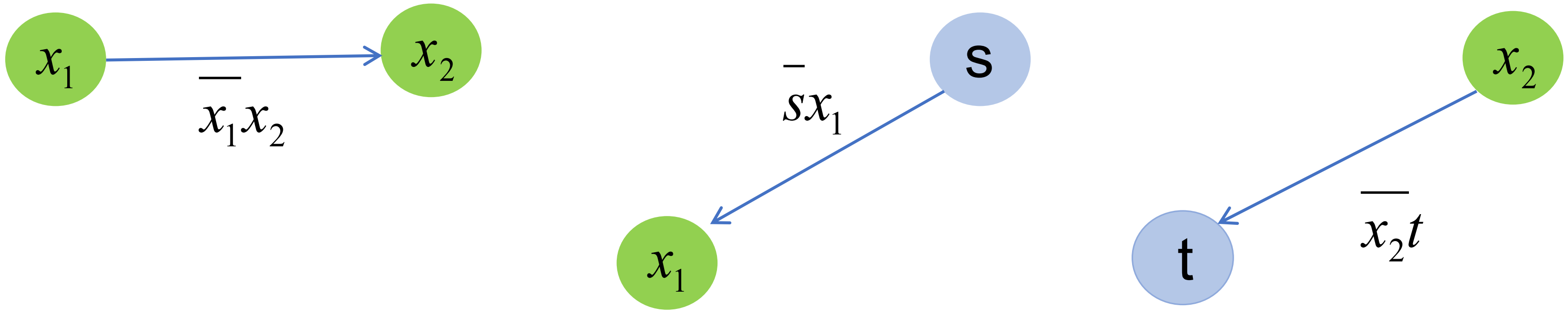
edge costs

source set

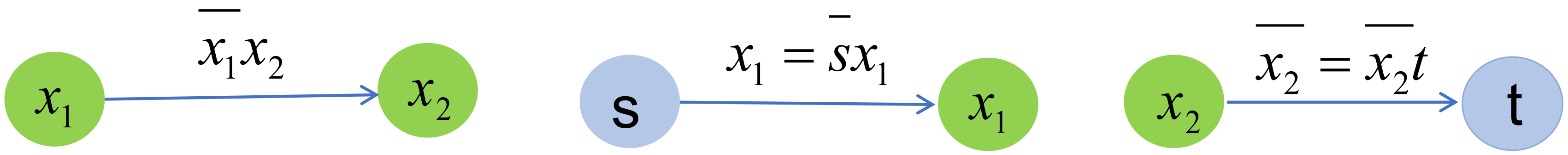
sink set

Network model for submodular QPBF

- A submodular QPBF f can be associated with a network G_v .
- There is 1-1 correspondence every edge in network and every term in f.
- Let us denote source by $s = 0$ and sink by $t = 1$.
- An edge that goes from x_1 to x_2 is denoted by $\overline{x_1 x_2}$.



Network model for submodular QPBF



- Given a QPBF we rewrite it using a posiform representation using only three types of terms: $\bar{x}_i x_j, \quad x_i, \quad \bar{x}_i,$

$$f = 3x_1 + x_2 - 4x_1x_2$$

$$f = 3x_1 + x_2 + (-4x_1x_2 + 4x_2 - 4x_2)$$

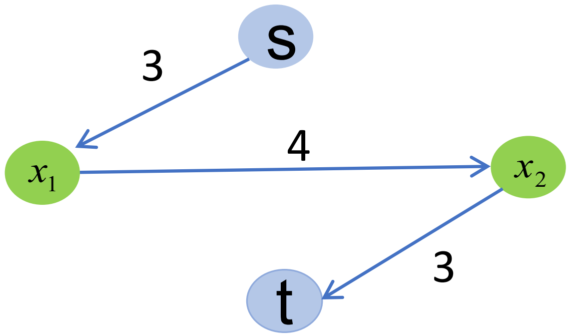
$$f = 3x_1 + x_2 + 4(1 - x_1)x_2 - 4x_2$$

$$f = 3x_1 - 3x_2 + 4\bar{x}_1 x_2$$

$$f = 3x_1 + (-3x_2 + 3 - 3) + 4\bar{x}_1 x_2$$

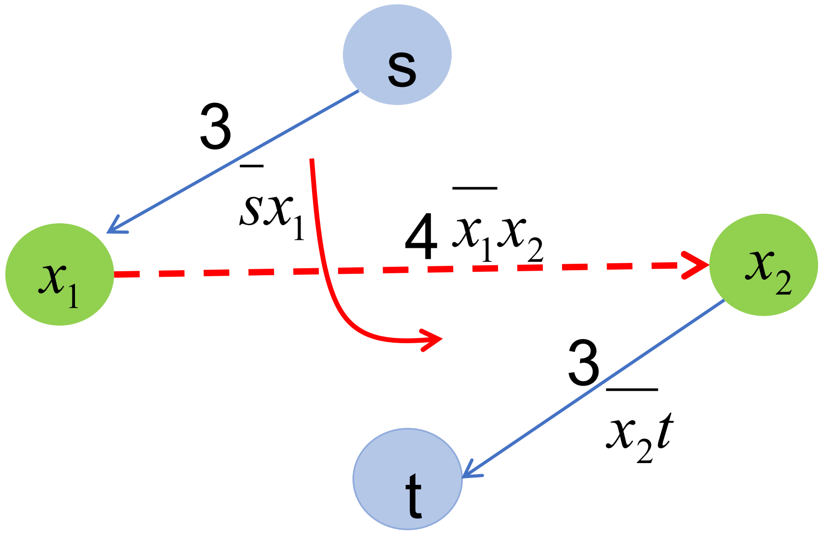
$$f = -3 + 3x_1 + 3(1 - x_2) + 4\bar{x}_1 x_2$$

$$f = -3 + 3\bar{s}x_1 + 3\bar{x}_2 t + 4\bar{x}_1 x_2$$



Network model for submodular QPBF

- There is a one-one correspondence between values of f and s-t cut values of G_v. [Hammer 1965]

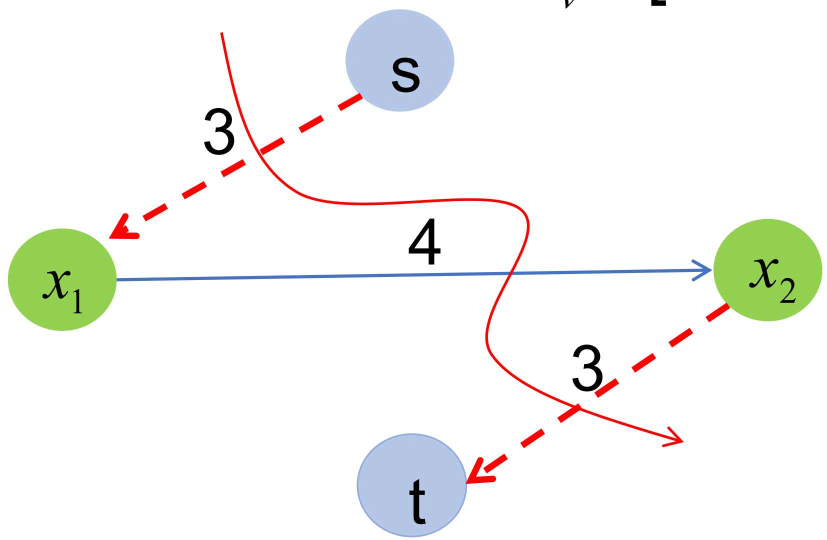


$$f(x_1 = 0, x_2 = 1) =$$
$$C(\{x_1, x_2\}) = 4$$

s-t mincut
[Ford&Fulkerson'62, Goldberg&Tarzan86]

Network model for submodular QPBF

- There is a one-one correspondence between values of f and s-t cut values of G_v. [Hammer 1965]



$$f(x_1 = 1, x_2 = 0) =$$

$$C(\{x_2, s\}, \{x_1, t\}) = 3 + 3 = 6$$

Thus we can compute the minimum of f using maxflow/mincut algorithm on the associated G_v.

s-t mincut [Ford&Fulkerson'62, Goldberg&Tarzan86]

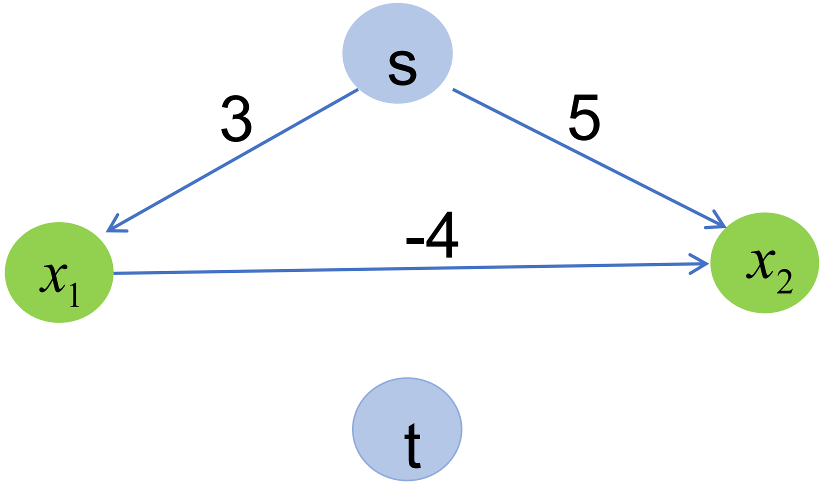
Network model for non-submodular QPBF

- A non-submodular QBPF f can be associated with a network G_v as follows:

$$f = 3x_1 + x_2 + 4x_1x_2$$

$$f = 3x_1 + 5x_2 - 4(1 - x_1)x_2$$

$$f = 3\bar{s}x_1 + 5\bar{s}x_2 - 4\overline{x_1}x_2$$



- There is no polynomial-time algorithm for s-t mincut on a network with negative edge capacities.
- A submodular QBPF can always be associated with a network with non-negative edge capacities.

Minimizing Quadratic Pseudo Boolean Functions

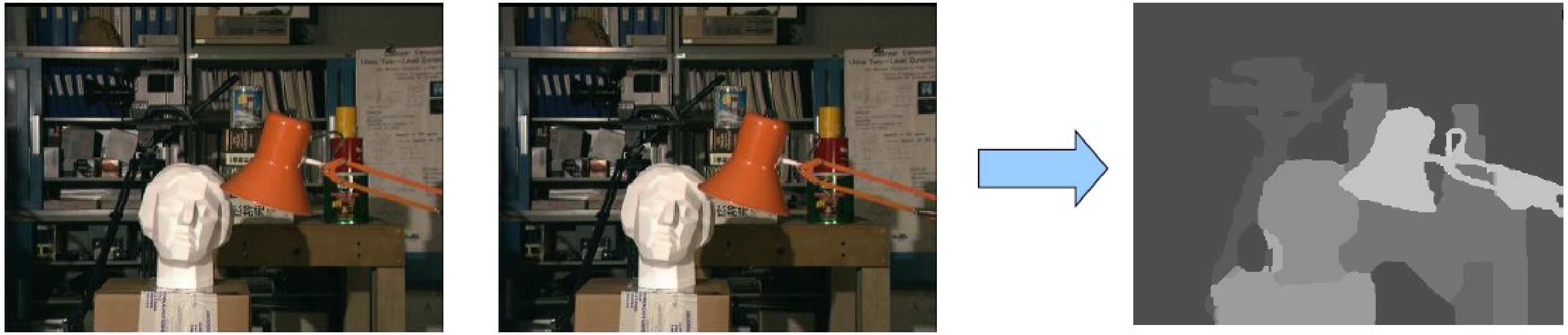
- If QPBF is submodular, use maxflow algo..

[Ford&Fulkerson’62,
Goldberg&Tarzan86]

- If QPBF is non-submodular, Belief propagation or other message passing algorithms.

[Boros&Hammer’2002]

Multi-label Problems



Left Camera Image Right Camera Image Dense Stereo Result

- Choose the disparities from the discrete set: $(1,2,\ldots,L)$

Multi-label Problems

Exact Methods:

Transform the given multi-label problems to Boolean problems and solve them using maxflow/mincut algorithms or QPBO techniques.

[Not covered in this course!]

Approximate Methods:

Develop iterative move-making algorithms where each move corresponds to a Boolean problem.

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Boolean Energy Function

- Variables $x_1, x_2, \ldots, x_n \in \{0,1\}.$

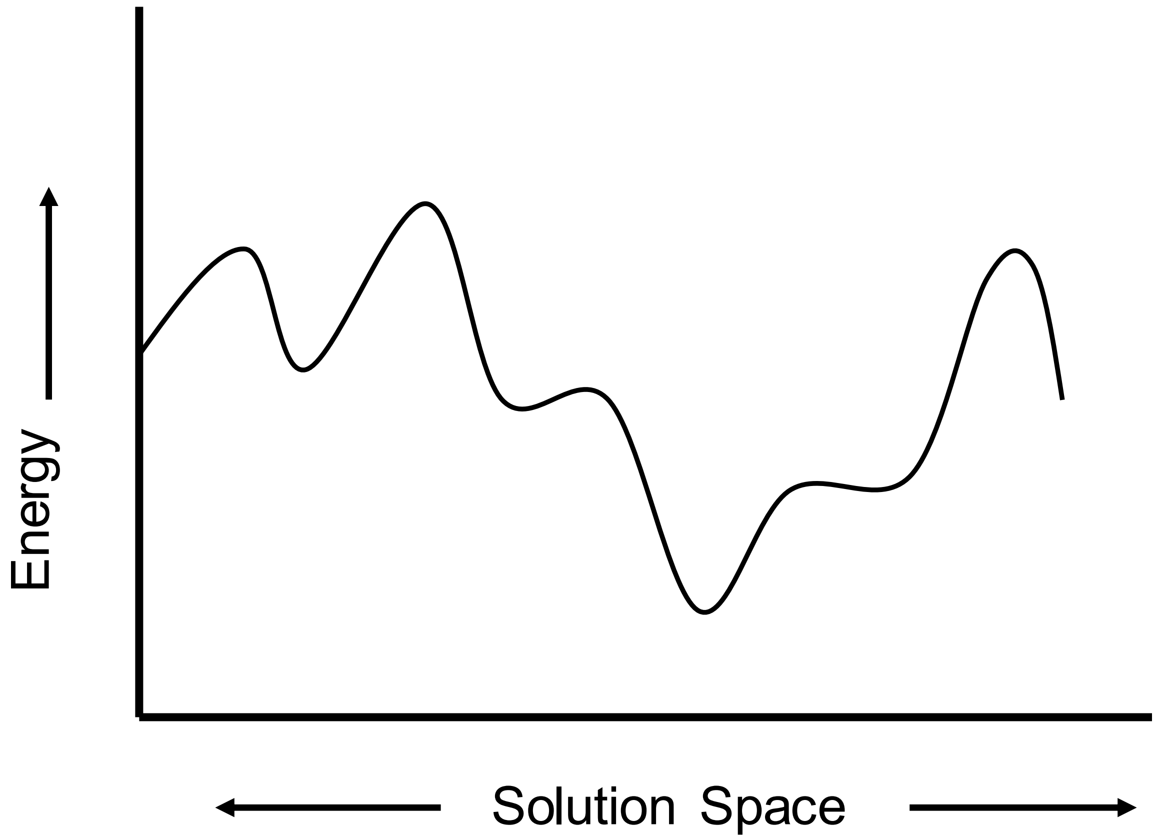
$\theta_{x_i}^{j}$ - cost of assigning $x_i = j \in \{0,1\}$.

$\theta_{x_i x_j}^{lm}$ - cost of jointly assigning $x_i = l$ and $x_j = m.$

Energy function:

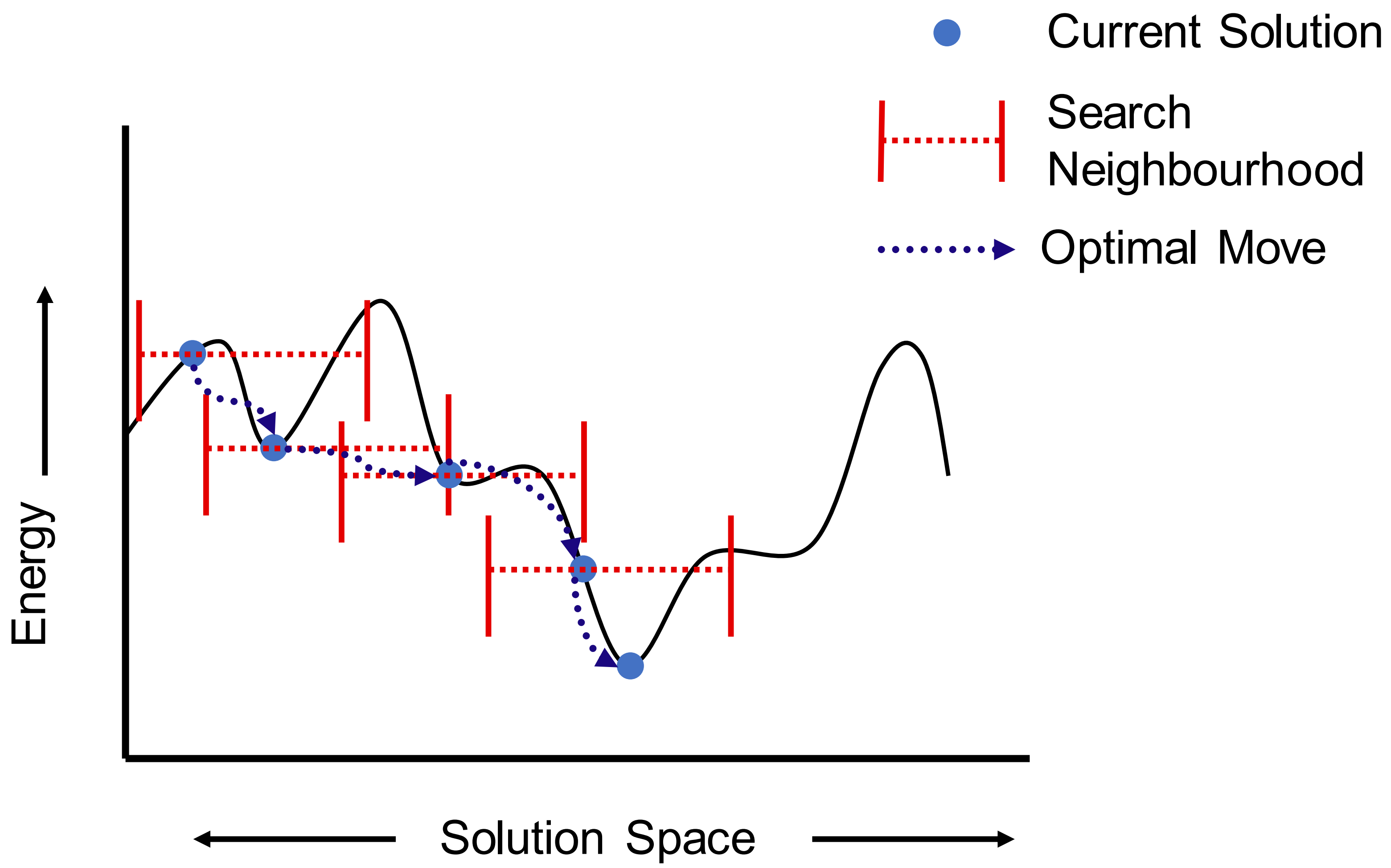
$$E(x_1, x_2) = \sum_{j=0}^{1} \theta_{x_1}^{j} \delta_{x_1}^{j} + \sum_{j=0}^{1} \theta_{x_2}^{j} \delta_{x_2}^{j} + \sum_{i=0}^{1} \sum_{j=0}^{1} \theta_{x_1 x_2}^{ij} \delta_{x_1}^{i} \delta_{x_2}^{j}$$

Move Making Algorithms



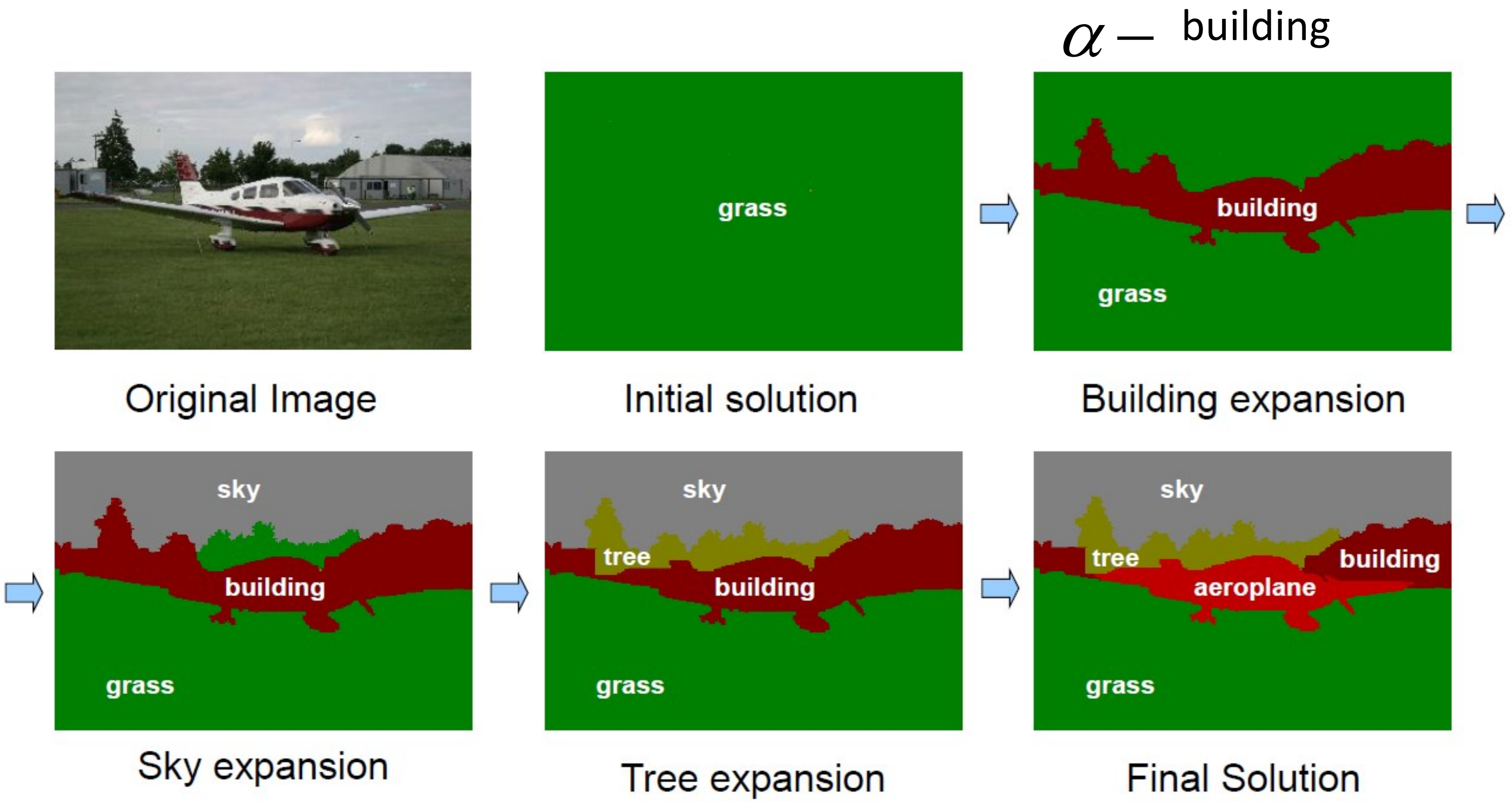
[Image courtesy: Pushmeet Kohli, Phil Torr]

Move Making Algorithms



[Image courtesy: Pushmeet Kohli, Phil Torr]

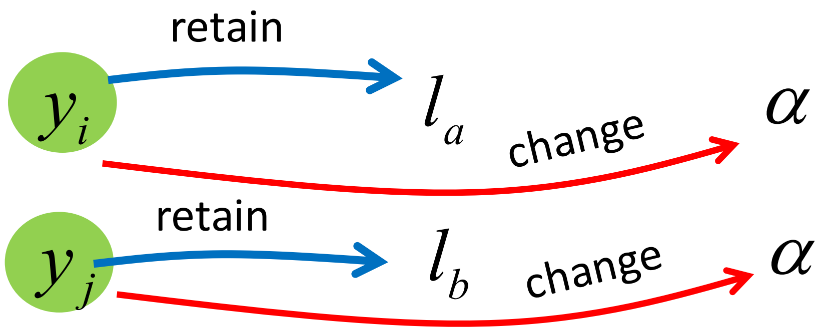
$\alpha -$ Expansion



[Boykov et al. 2001]

$\alpha -$ Expansion

- Let y_i and y_j be two adjacent variables whose labels are not α.



In the move space, we compute if the two variables should retain the same labels or move to label α.

[Boykov et al. 2001]

$\alpha -$ Expansion

- In the move space, we use two Boolean variables x_i and x_j to denote y_i and y_j respectively. The encoding is shown below:

$$y_i = l_a \Leftrightarrow x_i = 0 \qquad y_j = l_a \Leftrightarrow x_j = 0$$

$$y_i = \alpha \Leftrightarrow x_i = 1 \qquad y_j = \alpha \Leftrightarrow x_j = 1$$

- Submodularity condition states that the sum of main diagonal elements is less than the sum of elements in the off-diagonal:

$$\begin{bmatrix} \theta_{x_i x_j}^{00} & \theta_{x_i x_j}^{01} \\ \theta_{x_i x_j}^{10} & \theta_{x_i x_j}^{11} \end{bmatrix} = \begin{bmatrix} \theta_{y_i y_j}^{l_a l_b} & \theta_{y_i y_j}^{l_a \alpha} \\ \theta_{y_i y_j}^{\alpha l_b} & \theta_{y_i y_j}^{\alpha \alpha} \end{bmatrix}$$

[Boykov et al. 2001]

$\alpha -$ Expansion

- Submodularity condition states that the sum of main diagonal elements is less than the sum of elements in the off-diagonal:

$\theta^{00}_{x_i x_j}$	$\theta^{01}_{x_i x_j}$
$\theta^{10}_{x_i x_j}$	$\theta^{11}_{x_i x_j}$

=

$\theta^{l_a l_b}_{y_i y_j}$	$\theta^{l_a \alpha}_{y_i y_j}$
$\theta^{\alpha l_b}_{y_i y_j}$	$\theta^{\alpha\alpha}_{y_i y_j}$

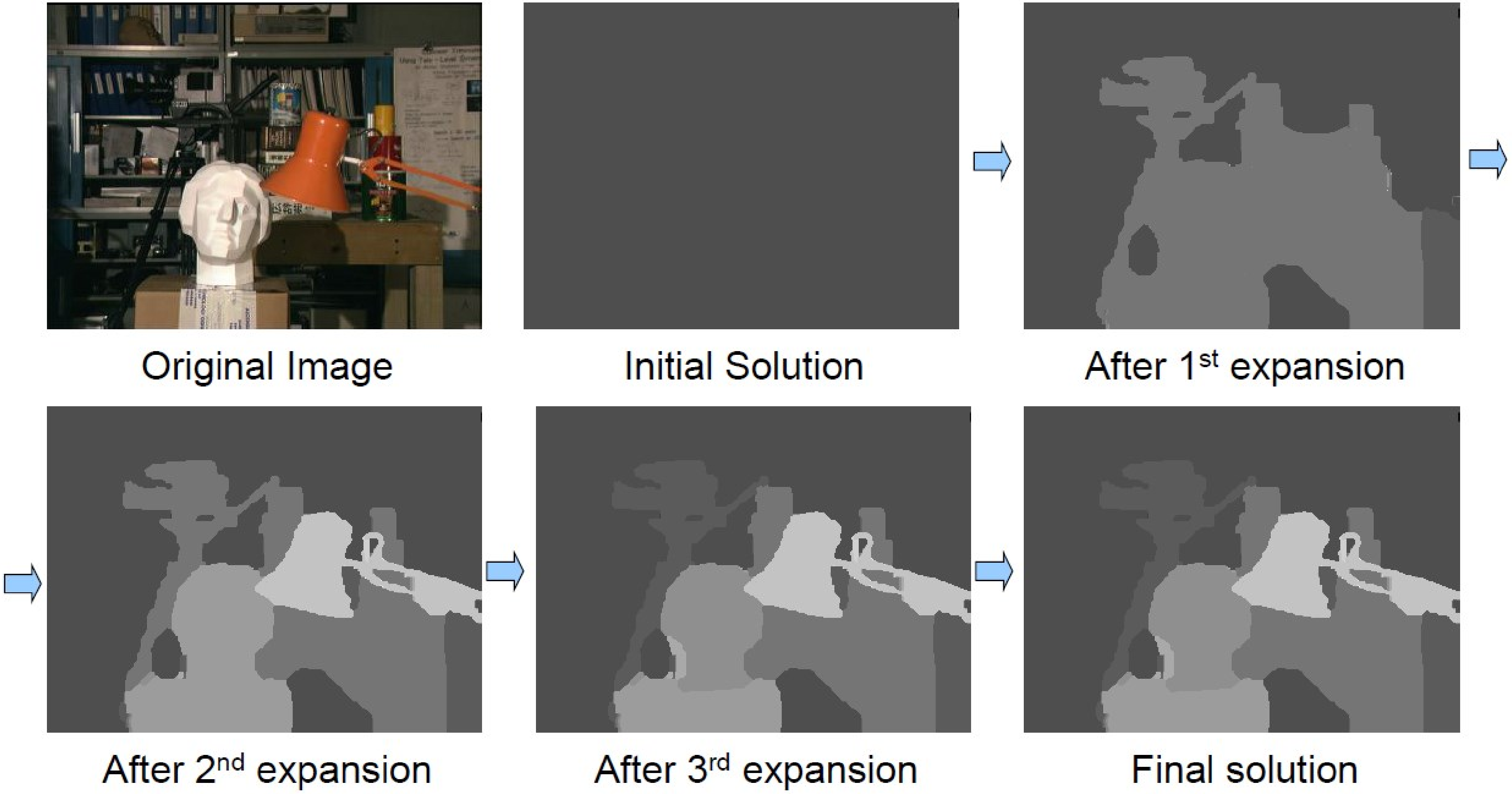
$$\theta^{l_a l_b}_{y_i y_j} + \theta^{\alpha\alpha}_{y_i y_j} - \theta^{l_a \alpha}_{y_i y_j} - \theta^{\alpha l_b}_{y_i y_j} \le 0$$

If the multi-label potentials satisfy metric condition:

$$\forall l_a, l_b \in L,$$
$$\theta^{l_a l_a}_{y_1 y_2} = 0,$$
$$\theta^{l_a l_b}_{y_1 y_2} = \theta^{l_b l_a}_{y_1 y_2} \ge 0,$$
$$\theta^{l_a l_b}_{y_1 y_2} + \theta^{l_b l_c}_{y_1 y_2} \ge \theta^{l_a l_c}_{y_1 y_2}$$

[Boykov et al. 2001]

$\alpha -$ Expansion



[Image courtesy: Lubor Ladicky]

[Boykov et al. 2001]

Thank You