

Multistage Methods II: RK Stability

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1 Introduction

We will now quickly review the stability properties of RK methods using the Butcher Tableau coefficients.

2 Stability

Let $\mathbf{b}^T = [b_1, b_2, \dots, b_s]$, $C = I[c_1, \dots, c_s]^T$, and $A = a_{ij}, i, j = 1, \dots, s$ (the Runge-Kutta matrix). Then, on the test ODE $y' = \lambda y$, we can show that any RK method yields

$$y^{n+1} = R(\Delta t \lambda) y^n, \quad (1)$$

$$R(\mu) = 1 + \mu \mathbf{b}^T (I - \mu A)^{-1} \mathbf{1}, \quad (2)$$

where $\mathbf{1} = [1, 1, \dots, 1]^T$ is an s -long vector of ones. As usual, for stability, we will require that

$$|R(\Delta t \lambda)| \leq 1. \quad (3)$$

2.1 A-Stability

If the region of absolute stability contains the entire left half of the complex plane, the RK method is A-stable.

2.2 L-Stability

An implicit, A-stable RK method has stiff decay if

$$\lim_{\operatorname{Re}(\mu) \rightarrow -\infty} R(\mu) = 0. \quad (4)$$

Such RK methods are called L-stable methods.

2.3 B-Stability

Often, the problems we solve are nonlinear. We'd like to extend the notion of stability to these problems as well.

First, we say that a function is *dissipative* if $\langle f(t, y) - f(t, z), (y - z) \rangle \leq 0$ for all y and z .

An ODE with a dissipative right-hand side f is *contractive*. In other words, for the ODE, $\|y(t) - z(t)\| \leq \|y(s) - z(s)\|$ for every pair of solutions y and z when $t \geq s$.

Armed with these definitions, we can say that a numerical method is *B-Stable* if every pair of numerical solutions u and v satisfy $\|u^{n+1} - v^{n+1}\| \leq \|u^n - v^n\|$ for all n , when solving a contractive ODE IVP. A B-Stable method is also A-stable.

Let $B = I\mathbf{b}$, and $M = BA + A^T B^T - \mathbf{b}\mathbf{b}^T$. An RK method is said to be *algebraically stable* if M is nonnegative semidefinite; *i.e.*, $x^T M x \geq 0$ for any x .

Finally, an algebraically stable RK method is B-stable.