

Linear Multistep Methods II: Consistency, Stability and Convergence

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1 Introduction

In the previous chapter, we presented some of the important linear multistep methods. In this chapter, we will discuss the consistency, stability and convergence of these methods by analyzing their coefficients α and β . Over the course of this discussion, we will explore the question of when to use explicit vs implicit methods.

2 Consistency, LTE and global error

A multistep method is called consistent if it is of order $s \geq 1$. This amounts to the following conditions on the coefficients:

$$\sum_{k=0}^s a_k = 0, \tag{1}$$

$$\sum_{k=0}^s k a_k = \sum_{k=0}^s b_k. \tag{2}$$

“Consistency” is simply the property of the numerical method accurately representing the differential equation it is discretizing.

This definition of consistency has an automatic consequence: for the global error to be ≥ 1 , we know the LTE must be smaller than $O(\Delta t)$. Thus, a numerical method must have an LTE that vanishes sufficiently fast.

However, consistency does not automatically imply convergence.

3 Stability

An important requirement for convergence is stability. Often, in real-world scenarios, we favor a lower-order numerical method over a higher-order one if it is more stable, for some appropriate definition of stability. This is typically the reason for selecting implicit methods over explicit methods. More on that in a moment.

3.1 Zero-Stability

Consider the general form of a multistep method, as written earlier:

$$\sum_{k=0}^s \alpha_k \mathbf{y}_{n+1-k} = \Delta t \sum_{k=0}^s \beta_k \mathbf{f}_{n+1-k}. \quad (3)$$

A multistep method is said to be *zero-stable* if the numerical solution it generates remains bounded as $\Delta t \rightarrow 0$ for finite final time T . That is, we require that all solutions to

$$\sum_{k=0}^s \alpha_k \mathbf{y}_{n+1-k} = 0 \quad (4)$$

remain bounded as $n \rightarrow \infty$. To show this, we assume the solution to the recurrence has the form

$$\alpha_n = \alpha_0 z^n, \quad (5)$$

where z^n is z to the power n , and z may be a complex number. The *characteristic polynomial* $\rho(z)$ is said to obey the *root condition* if

1. If all its zeros lie within the closed unit disk, *i.e.*, $|z| \leq 1$, and
2. If $|z| = 1$, $\rho'(z) \neq 0$ (no repeated roots on the unit circle).

If the characteristic polynomial obeys the root condition, the linear multistep method is automatically zero-stable; the root condition is a way of ensuring boundedness of the solutions to (4).

3.2 Absolute Stability

We could now ask ourselves what happens to the numerical solution as $t \rightarrow \infty$ for a fixed Δt . This view of stability is called *absolute stability*.

Consider the model problem

$$y'(t) = \lambda y(t), t \in [t_0, T], \quad (6)$$

$$y(t_0) = y_0. \quad (7)$$

We know the solution is given by $y(t) = y_0 e^{\lambda t}$. If we perturb the initial condition by some small number ε , we have a solution of the form $y(t) = (y_0 + \varepsilon) e^{\lambda t}$.

Clearly, for real λ , if $\lambda > 0$, large changes in the solution occur even for small perturbations ε ; the ODE itself is unstable! Thus, we will restrict our attention to discretizations of problems with $\lambda \leq 0$.

Forward Euler

We will now test the stability of Forward Euler on the model problem. We have

$$y_{n+1} = y_n + \Delta t f(t_n, y_n), \quad (8)$$

$$= y_n + \Delta t \lambda y_n, \quad (9)$$

$$= (1 + \lambda \Delta t) y_n. \quad (10)$$

By induction, then, $y_n = (1 + \lambda \Delta t)^n y_0$. Since the model problem has a decaying solution for $\lambda < 0$, we would like Forward Euler to have the same property. In other words, we want the *growth factor* $|1 + \lambda \Delta t| < 1$. Clearly, Forward Euler is absolutely stable only if Δt is sufficiently small; in other words, it is conditionally stable.

The set of $\lambda \Delta t$ values for which the growth factor is less than one is called the *region of absolute stability*. In a later subsection, we will see some of these regions for different multistep methods. In general, λ is complex; in fact, in the PDE context, it is typically an eigenvalue of some discretized differential operator.

3.3 A-stability

This leads us to the concept of *A-stability*, not to be confused with absolute stability. A numerical method is *A-stable* if its region of absolute stability include the entire complex half-plane with negative real part.

In your assignment, you will show that Backward Euler is A-stable.

3.4 Some Stability Regions

Let us look at some regions of absolute stability for AB, AM, AB-AM and BDF methods. From Fig. 1, it is clear that the higher the order of the Adams-Bashforth method, the smaller its region of stability. This means that as we increase the order of the AB method, we must take smaller and smaller time-steps. While this is not necessarily an issue from the perspective of truncation error (why?), it means that the AB methods are inapplicable to many problems where the λ values lie outside the region of stability. The situation is a little better for AM methods, but not much better. Fig. 2 shows that AM3 has a

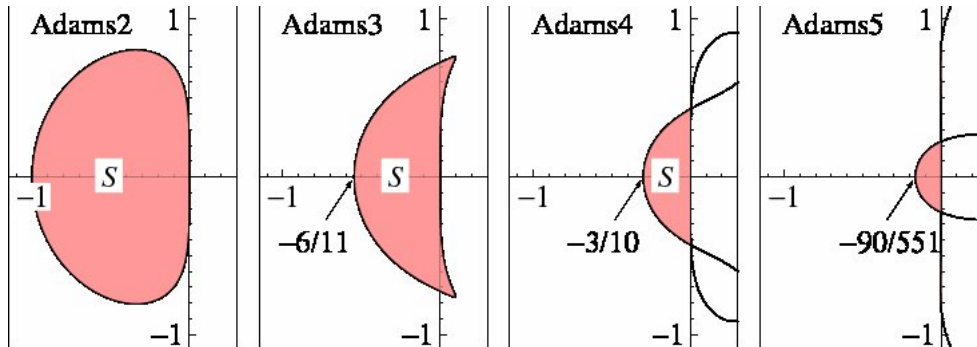


Figure 1: Stability regions for AB methods

much larger stability region than AB3. However, this figure also shows the issue with predictor-corrector methods; as you can guess, using an explicit method in conjunction with an implicit method shrinks the resulting stability region to something halfway between the two methods. What about the BDF methods?

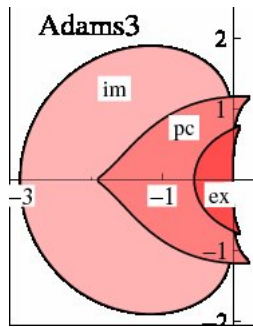


Figure 2: Stability Regions for AB, AM and predictor-correct AB-AM methods

The stability diagrams are shown in Fig. 3. Here, the stability regions are indicated in *pink*! Notice how stable the BDF methods are! This means that, when using these methods, we are allowed to generally take large time-steps. Notice, however, that the methods have smaller stability regions as we increase their order. This pattern is true for all multistep methods.

3.5 Implicit or Explicit?

In practice, we use these stability diagrams to tell us whether we ought to use an implicit method or an explicit method. Implicit methods are far more stable than explicit methods on problems where the eigenvalues have very small imaginary parts. However, there is an extra cost associated with an implicit

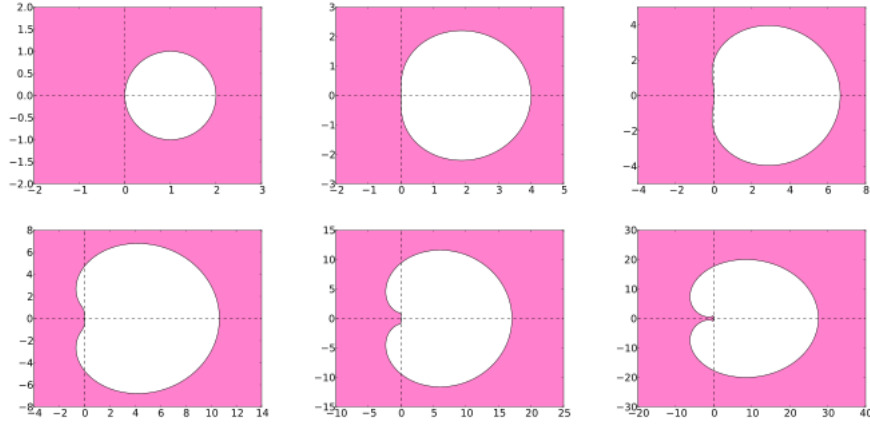


Figure 3: Stability regions for BDF methods are indicated in pink.

method– we typically need to solve a linear or nonlinear system when using one.

In general, implicit methods are used to solve *stiff* problems.

3.5.1 What is a “Stiff” problem?

For our purposes, a linear problem is stiff if all its eigenvalues have negative real parts, and the ratio of the magnitudes of the real parts of largest and smallest eigenvalues is large. In this scenario, it is hard to beat the BDF methods due to their gigantic stiffness regions.

Practical advice: In practice, we pick a high order method that has a reasonable region of absolute stability, and select the largest time-step size Δt allowed by the stiffness diagrams. If the function f is nonlinear in y , we will often select an explicit method so as to avoid the solution of a nonlinear system of equations; we will live with the time-step restriction this imposes.

4 Convergence

We can now discuss convergence. An LMM is said to be *convergent* if as $\Delta t \rightarrow 0$, the numerical solution generated by the LMM approaches the exact solution.

4.1 The Dahlquist Equivalence Theorem

A linear multistep method is convergent if and only if it is consistent and zero-stable.

Consequently, if a linear multistep method's characteristic polynomial does not obey the root condition, the method cannot be convergent.

4.2 Dahlquist Barriers

The Dahlquist barriers establish limits on the convergence and stability properties of multistep methods. While we will not prove them here, they are definitely important.

4.2.1 Dahlquist's First Barrier

The maximal order of a convergent s -step method is at most

- $s + 2$ for implicit schemes with even s ,
- $s + 1$ for implicit schemes with odd s ,
- s for explicit schemes.

Note: It can be shown that implicit s -step methods of order $s + 2$ are not very stable.

4.2.2 Dahlquist's Second Barrier

There are no explicit A-stable linear multistep methods. Further, the implicit A-stable linear multistep methods have order of convergence at most $s = 2$.

4.3 Root condition failure for BDF methods

While all Adams methods satisfy the root condition and are therefore zero-stable, BDF methods satisfy the root condition only if $1 \leq s \leq 6$.