

2250-7 Practice Exam 2

Name and uID: _____

Concepts and skills required for exam

- Understand how to compose a set of linear equations into a matrix equation
- Know how the rules of matrix algebra, how to determine existence and compute matrix inverses, matrix multiplication, transpose, etcetera...
- Understand the definition of row equivalence and its meaning: If A and B are row equivalent in means $Ax = 0$ and $Bx = 0$ have the same solution.
- Know the algebraic properties of a vector space and subspace, and be able to determine if a set is a subspace or vector space.
- Be able to (a) transform a matrix to reduced row-echelon form and (b) interpret the dimension of the dimension of the homogeneous subspace, and (c) find a basis set that spans the subspace.
- As per above, know the definition of linear independence and dependence and be able to determine it for a set of vectors $\{v_1, v_2, \dots, v_k\}$.
- Know the definition of a basis and span for a vector space, and how to determine the dimension of a vector space
- Know the definition of the Wronskian and how to use it to determine linear independence of a set of functions.
- Know the definition of a linear differential operator L , how to determine if an operator is linear, what order, etc.
- Know how to find the eigenvalues and eigenvectors of a linear differential equation with constant coefficients $Lx = 0$.
- Know the definition of the multiplicity k of eigenvalues and how to obtain linear independent spanning sets for the k -dimensional subspace for a λ .
- Know the definition of spring-mass-dashpot second-order linear system, how to determine if its underdamped, over damped, or critically damped.
- Be able to find the homogeneous or complementary solution describing free motion of a mass-spring-dashpot system given initial position and velocity.
- Be able to solve for amplitude and phase delay of a periodically forced mass-spring-dashpot system using the method of undetermined coefficients and trigonometry. phase delay formula will be provided.

- Understand resonance for forced equations.
- Given a general constant coefficient $Lx = F$, with F being of exponential, sinusoidal, or polynomial type, or products of all three, be able to use the method of undetermined coefficients to solve for particular solutions.
- Given $Lx = F$, be able to find the complementary (homogeneous) solution, the particular solution, and then finding the correct weightings $c_1, c_2, \dots$ to solve for the particular solution to an IVP.
- Know the definition of the Laplace transform and how to compute $F(s)$ from the definition, and determine conditions on s for existence.

Write your answer in the space provided. Show work for full credit. Unless otherwise noted, y is the dependent variable, x is the independent variable, and $y' = \frac{dy}{dx}$. IVP = initial value problem, DE = differential equation.

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -2 & 0 & 1 \\ 3 & -6 & -5 & 1 & 4 \\ -2 & 4 & 2 & -2 & 4 \end{bmatrix}$$

and its row reduced echelon form

$$rref(\mathbf{A}) = \begin{bmatrix} 1 & -2 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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- (a) Find the general solution to $\mathbf{Ax} = \mathbf{0}$. Write your answer as a linear combination.
- (b) What is the dimension of the solution space ?

Solution:

(a)

$$\begin{matrix} & x_5 = r, & x_4 = s, & x_2 = t \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} & = & \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} r + & \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + & \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t \end{matrix}$$

- (b) What is the dimension of the solution space? Three!

2. Suppose $\mathbf{A}$ is a 3×3 matrix and the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

span the subspace defined by $\mathbf{Ax} = \mathbf{0}$. Determine if the vector

$$\mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

is in $\text{Span}(\mathbf{u}, \mathbf{v})$. Will $\mathbf{Aw} = \mathbf{0}$?

Solution:

$$\text{Det} \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = -1 - 2(2 - 1) - 1(-1) = -1 - 2 + 1 = -2 \neq 0$$

So the columns are linearly independent, which means that $\mathbf{w}$ cannot be made equal to any linear combination of the other vectors.

3. Do the functions $y_1(x) = e^{2x}$, $y_2(x) = 1$ form a basis for the solution space of the DE $y'''(x) - 2y''(x) = 0$? Justify your answer.

Solution: Quick answer: this is a third-order linear DE, so there must be three LI solution functions in order to span the solution space, but only two are presented, so there must be one more that is missing.

Long answer:

$$\begin{aligned} r^3 - 2r^2 &= r^2(r - 2) = 0 \\ \implies r &= 0, 0, 2 \end{aligned}$$

There is a repeated root so $\text{span}\{1, t, e^{2t}\}$ span the solution space.

4. Suppose the functions $y_1(t) = e^{-2t}$ and $y_2(t) = e^{2t}$ span the solution space for the homogeneous linear DE:

$$L(y) = 0$$

. Specify the form of the solution to the non-homogeneous equation $L(y) = e^{2t}$

Solution: The right hand side of the non homogeneous problem is in the span of a one-dimensional homogeneous solution subspace e^{2t} , corresponding to a root $r = 2$ of multiplicity one. Hence, the form of the particular solution is $y_p = Ate^{2t}$.

5. Find the general solution of the undamped forced oscillator

$$x''(t) + 4x(t) = 10 \cos(3t).$$

Solution: $x(t) = c_1 \sin(2t) + c_2 \cos(2t) - 2 \cos(3t)$

6. Find the general solution of the DE

$$y'' - y' - 2y = xe^{2x}.$$

Solution: $y(x) = c_1 e^{2x} + c_2 e^{-x} + (1/18)x(3x - 2)e^{2x}$

7. Given the matrix $\mathbf{A} = \begin{bmatrix} -1 & 3 & 0 \\ -1 & 2 & 1 \\ -3 & 5 & 4 \end{bmatrix}$.

(a) Find $\det(\mathbf{A})$.

(b) What does $\det(\mathbf{A})$ tell you about the number of solutions to $\mathbf{Ax} = \mathbf{0}$?

Solution:

$$\det(\mathbf{A}) = -1(8 - 5) - 3(-4 + 3) = -3 + 3 = 0$$

the zero determinant means that the solution space of $\mathbf{Ax} = \mathbf{0}$ is at least one dimensional. If $\det(\mathbf{A}) \neq 0$, then the only solution is $\mathbf{x} = \mathbf{0}$.

8. Determine if the vectors are linearly independent or not (Note: you can't use a determinant on a matrix formed by these vectors since they are 4D and there are only 3 of them so the matrix would not be square. Instead, use the primary definition of linear independence/dependence).

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: Without computing any matrix operations, it's clear that $\mathbf{u}$ and $\mathbf{v}$ are negatives of each other, so they are not linearly independent.

9. Find the general solution to the DE

$$y^{(6)}(x) + 8y^{(3)}(x) + 16y(x) = 0.$$

Solution: Find characteristic equation

$$r^6 + 8r^3 + 16 = (r^3 + 4)(r^3 + 4)$$

$$\text{let } r = -4^{\frac{1}{3}}e^{i\theta}$$

$$\text{then } r^3 = -4e^{i3\theta}$$

$$\text{where } 3\theta = 2\pi n, \quad n = 0, 1, 2, \dots$$

$$\text{then } \theta = \frac{2}{3}\pi n \quad n = 0, 1, 2$$

so there are three complex roots, each with multiplicity 2:

$$r_n = -4^{\frac{1}{3}}\left(\cos\left(\frac{2}{3}\pi n\right) + i\sin\left(\frac{2}{3}\pi n\right)\right)$$

$$y = c_0e^{r_0t} + c_1e^{r_1t} + c_2e^{r_2t} + t(b_0e^{r_0t} + b_1e^{r_1t} + b_2e^{r_2t})$$

Clearly, this problem requires the tools of complex variables to solve, so you need not worry about something like this on the exam.

10. Suppose we have the DE $y^{(10)}(t) = 0$

- (a) Find a basis for the solution space of the homogeneous DE
 (b) Compute the particular solution to the non-homogeneous DE $y^{(10)}(t) = t^{11}$

Solution: Find characteristic equation

$$r^{10} = 0$$

$\implies r = 0$ with multiplicity 10. The basis for the solution space is ten-dimensional:

$$\{1, t, t^2, \dots, t^9\}$$

To get the particular, we try At^{11+10} , whose tenth derivative is

$$A(21)(20)(19)(18)(17)(16)(15)(14)(13)(12)t^{11}$$

So $A = ((21)(20)(19)(18)(17)(16)(15)(14)(13)(12))^{-1}$.

11. Consider the DE

$$y'''(x) + y'(x) = 0.$$

(a) Verify that $y_1(x) = 1$, $y_2(x) = \cos(x)$ and $y_3(x) = \sin(x)$ are solutions of the DE.

(b) Verify by calculating the Wronskians that y_1, y_2, y_3 are linearly independent.

Solution:

(a) Yes!

(b)

$$\begin{aligned} & \text{Det} \begin{bmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{bmatrix} \\ &= 1 \times \text{Det} \begin{bmatrix} -\sin(x) & \cos(x) \\ -\cos(x) & -\sin(x) \end{bmatrix} \\ &= \sin^2(x) + \cos^2(x) = 1 \neq 0 \end{aligned}$$

Independence!

12. Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

Solution:

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{4}{3} & -\frac{1}{3} & -1 \\ -\frac{4}{3} & \frac{1}{3} & 2 \end{bmatrix}$$

13. Given the vectors and matrices defined below

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Compute the following matrix operations

(a) $\mathbf{u}^T \mathbf{w} =$

(b) $\mathbf{A}\mathbf{u} =$

(c) $\mathbf{w}^T \mathbf{A} =$

Solution:

(a) $\mathbf{u}^T \mathbf{w} = -1$

(b)

$$\mathbf{A} \mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

(c)

$$\mathbf{w}^T \mathbf{A} = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$$

14. Compute the Laplace transform of the function

$$f(t) = \begin{cases} 1 - \frac{1}{2}t, & 0 \leq t \leq 2 \\ 0, & \text{else} \end{cases}$$

Solution:

$$\begin{aligned} \mathcal{L}(f) &= \int_0^2 \left(1 - \frac{1}{2}t\right) e^{-st} dt \\ &= -\frac{1 - 2s - e^{-2s}}{2s^2} \end{aligned}$$

15. Solve the homogeneous IVP: $y''' - 8y' = 0$ with $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$.

Solution:

$$y(t) = \frac{18}{16} - \frac{1}{16}(e^{2\sqrt{2}t} + e^{-2\sqrt{2}t})$$

16. Solve for the general solution of $y'' + 2y' + 2y = e^{-x} \sin(x)$.

Solution:

$$y(t) = e^{-t}(c_1 \cos(t) + c_2 \sin(t)) - \frac{t}{2} e^{-t} \cos(t)$$