

2250-7 Midterm Exam 2

Name and UnID: _____

Write your answer in the space provided. Show work and reasoning for full credit. Unless otherwise noted, x is the dependent variable, t is the independent variable, and $x' = \frac{dx}{dt}$. IVP = initial value problem, DE = differential equation, and vector and matrix quantities will be expressed in **bold** notation: $\mathbf{v}$. Each question is worth 5 points unless otherwise indicated.

1. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find a basis set $\{\mathbf{v}_1, \mathbf{v}_2, \dots\}$ that spans the solution space S defined as all vectors $\mathbf{x} \in \mathbb{R}^4$ that satisfy $\mathbf{A}\mathbf{x} = \mathbf{0}$; and, what is the dimension of S ?

Solution:

(a)

$$\begin{aligned} & x_4 = s, \quad x_2 = t \\ \implies & x_3 = -2s, \quad x_1 = x_2 - 2x_3 - x_4 = t - 2(-2s) - s = t - 5s \\ & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -2 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} t \end{aligned}$$

(b) What is the dimension of the solution space? Two!

2. Suppose $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for a subspace W of $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Determine if the vector

$$\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

is an element of $\text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$. To answer this question, set up the appropriate equations, solve them, and make your assertion based on your results. Credit will be given based on how you perform and justify your methods, not what you conclude. More specifically:

- (a) State explicitly what it means for “ $\mathbf{w} \in \text{span}(\{\mathbf{v}_1, \mathbf{v}_2\})$ ”.

- (b) Use the appropriate linear algebra methods to determine if (a) is satisfied.
3. (a) (3 points) Show that the unit sphere, defined as the set of all vectors (x, y, z) that satisfy $x^2 + y^2 + z^2 = 1$ is not a *vector subspace* of $\mathbb{R}^3$. Clearly present your reasoning *and* justify your reasoning with an example.

Solution: Pick two vectors $\mathbf{u}$ and $\mathbf{v}$ that satisfy the sphere equation. Add them, and you can easily translate off the sphere. Similarly, scale one of those vectors by a constant, and you can again end up off the sphere. The sphere is a subset of $\mathbb{R}^3$, but not a vector subspace.

- (b) (2 points) Let S be the subspace of $\mathbb{R}^4$ defined by $x_1 = 2x_2$ and $x_3 = 5x_4$. Find a basis for S . Hint: Try plugging in numbers; the basis is extremely easy to find.

Solution: $(2, 1, 0, 0)$ and $(0, 0, 5, 1)$ form a basis for S . Any scaled versions are also acceptable.

4. Find a basis for the solution space of the DE

$$x''' + 4x'' + 4x' = 0$$

Solution:

$$\begin{aligned}\lambda^3 + 4\lambda^2 + 4\lambda &= 0 \\ \lambda(\lambda^2 + 2\lambda + 1) &= \lambda(\lambda + 2)^2 = 0 \\ \lambda_1 &= 0, \lambda_1 = -2, \lambda_2 = -2 \\ x(t) &= c_1 + c_2e^{-2t} + c_2te^{-2t}\end{aligned}$$

5. Find the free motion function $x(t)$ of a mass-spring-dashpot system with $m = 2$, $c = 0$, and $k = 8$, with $x(0) = 1$, $x'(0) = -1$. Additionally, indicate if the system underdamped, overdamped, or critically damped.

Solution:

$$\begin{aligned}2x'' + 8x &= 0 \\ x'' + 4x &= 0 \\ x(t) &= c_1 \cos(2t) + c_2 \sin(2t) \\ c_1 &= 1, c_2 = -1/2\end{aligned}$$

6. Find the particular solution $x_p(t)$ to the forced mass-spring-dashpot equation

$$x'' + 2x' + x = \sin(2t).$$

What is the maximum amplitude C of $x_p(t)$? The amplitude calculation for C can be left as an algebraic expression, but hint: it does simplify.

Solution:

$$\begin{aligned}x(t) &= A \cos(2t) + B \sin(2t) \\x' &= -2A \sin(2t) + 2B \cos(2t) \\x'' &= -4A \cos(2t) - 4B \sin(2t) \\-4A \cos(2t) - 4B \sin(2t) - 4A \sin(2t) + 4B \cos(2t) + A \cos(2t) + B \sin(2t) &= \sin(2t) \\-4B - 4A + B &= 1 \\-4A + 4B + A &= 0 \\\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\B &= 3/4A \\-16/4A - 9/4A &= 1 \\-25A/4 &= 1 \\A &= -4/25 \\B &= -3/25 \\C &= \sqrt{(4/25)^2 + (3/25)^2} = \sqrt{(16+9)/25^2} = 1/5\end{aligned}$$

7. Solve for the solution of $x'' + 4x' + 3x = e^{-t}$, with $x(0) = 1$, $x'(0) = 0$.

Solution:

$$\begin{aligned}\lambda^2 + 4\lambda + 3 &= (\lambda + 3)(\lambda + 1) = 0 \implies \lambda = -3, -1 \\x(t) &= c_1 e^{-t} + c_2 e^{-3t} + \frac{1}{2} t e^{-t} \\c_1 &= 5/4, c_2 = -1/4\end{aligned}$$

8. Find the general solution to the following DE:

$$x'' + 4x' + 4x = \cosh(t).$$

Solution: This can either be tackled by breaking the cosh function into a sum of two exponentials and solving, or by using the method of variation of parameters. The solution is $x = u_1 y_1 + u_2 y_2$, where $y_1 = e^{-2t}$, $y_2 = te^{-2t}$, $u_1 = -\frac{1}{18}e^x(9(x-1) + e^{2x}(3x-1)) + C_1$ and $u_2 = \frac{1}{6}e^x(3 + e^{2x}) + C_2$.

9. (5 points) (EXTRA CREDIT) Derive an expression for the solution of the following forced spring-mass system:

$$mx'' + kx = F_0 \sin(\omega t).$$

You MUST express the solution as the sum of a “wave” term and a second term corresponding to the forcing, as we did in class.

10. (5 points) (EXTRA CREDIT) Derive an expression for the solution of the following forced spring-mass system:

$$mx'' + kx = F_0 e^{i\omega t}.$$

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