

2250-7 Midterm 1 study guide and problems

Name and uID: _____

Write your answer in the space provided. Show work for full credit. Unless otherwise noted, y is the dependent variable, x is the independent variable, and $y' = \frac{dy}{dx}$. IVP = initial value problem, DE = differential equation.

1. Concepts, topics, and skills

- (a) What is a DE ?
- (b) What is the order of a DE ?
- (c) What is an IVP ?
- (d) How do you check whether a function solves a DE ? an IVP ?
- (e) What is a slope field or what is the connection between a slope field and a DE ?
- (f) What is an autonomous DE ?
- (g) What is an equilibrium solution to an autonomous DE ?
- (h) What is a general solution to a DE?
- (i) What is a particular solution to a DE?
- (j) What are the conditions for existence and uniqueness of solutions to a DE?
- (k) What is a system of linear equations ?
- (l) What does the last row(s) of the row reduced form of an augmented matrix $[Ab]$ tell you about the solutions of $Ax = b$.
- (m) What kind of solutions to you expect when solving a linear system of equations ?
- (n) Is $\mathbf{AB} = \mathbf{BA}$? Is $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$?
- (o) Be able recognize and solve DEs that are distinctively be solved by
 - direct integration
 - separation of variables.
 - first-order linear DE
- (p) Be able to compose and solve a linear system of equations and describe the solution via parameterization if necessary.

2. Without solving the DE, determine whether the solution to the IVP

$$yy' = x - 1, \quad y(0) = 1$$

exist and is unique.

HINT: Check the hypothesis of the existence and uniqueness theorem.

Solution:

$$y' = (x - 1)/y = f(x, y)$$
$$f_y(x, y) = -(x - 1)/y^2$$

f and f' are both continuous at $(0,1)$ thus the solution exists and is unique for some interval I containing $x = 0$.

3. **Without solving the DE**, determine whether the solution to the IVP

$$yy' = x - 1, \quad y(1) = 0$$

exist and is unique.

HINT: Check the hypothesis of the existence and uniqueness theorem.

Solution: From above, the y -derivative of f is discontinuous at $(1, 0)$, thus existence and uniqueness is not guaranteed. Using separation of variables, we can solve for a solution $y = \sqrt{x^2 - 2x + 1} = x - 1$; however, $y = 0$ is also a solution. Therefore, we see there are two solutions that both pass through $(1, 0)$ which verifies that uniqueness is not satisfied.

4. Show that $y(x) = c_1e^{2x} + c_2e^{-3x}$ solves the DE

$$y'' + y' - 6y = 0.$$

Solution:

$$y'(x) = 2c_1e^{2x} - 3c_2e^{-3x}$$
$$y''(x) = 4c_1e^{2x} + 9c_2e^{-3x}$$
$$4c_1e^{2x} + 9c_2e^{-3x} + 2c_1e^{2x} - 3c_2e^{-3x} - 6(c_1e^{2x} + c_2e^{-3x})$$
$$= c_1(4 + 2 - 6)e^{2x} + c_2(9 - 3 - 6)e^{-3x} = 0$$

5. Consider the DE

$$y' = x + \sin y.$$

A solution curve passes through the point $(1, \pi/2)$. What is the slope at that point ?

Solution:

$$y' = 1 + \sin(\pi/2) = 2$$

6. Consider the DE

$$y' = f(y).$$

Suppose $f(y)$ is continuous and differentiable and smooth for every y , and the following facts are known: $f(4) = 0$, $f(10) = 0$, $f'(4) = -1$, $f'(10) = 3$. There are no other zeros for f except at $y = 4, 10$. Describe the qualitative behavior of solutions $y(x)$ with initial values $y(0) = 2$, $y(0) = 5$, $y(0) = 20$?

Solution: The DE is autonomous, and $y = 4$ is a stable equilibrium, while $y = 10$ is an unstable equilibrium. $y_0 = 2, 5$ will head toward 4 while $y_0 = 20$ will go upward, away from $y = 10$.

7. Solve the DEs

(a) $y' = \cos(x)$;

(b) $y' + 2y = y^2$;

(c) $y' = 3yx^2$;

(d) $xy' = \frac{1}{y^3}$.

Solution:

(a) $y' = \cos(x)$;

$$y(x) = \sin(x) + C$$

(b) $y' + 2y = y^2;$

$$\begin{aligned}
 y' &= y^2 - 2y = y(y - 2) \\
 \frac{dy}{y(y - 2)} &= dx \\
 \frac{1}{y(y - 2)} &= \frac{A}{(y - 2)} + \frac{B}{y} \\
 1 &= Ay + B(y - 2) \\
 A = 1/2, \quad B &= -1/2 \\
 \int \frac{1}{(y - 2)} - \frac{1}{y} dy &= 2x + C \\
 \ln(|y - 2|) - \ln(|y|) &= 2x + C \\
 \ln(|y|/(|y - 2|)) &= -2x + C \\
 \frac{|y|}{|y - 2|} &= Ae^{-2x}
 \end{aligned}$$

The sign and value of A depends on which initial condition one chooses.

$$\begin{aligned}
 y &= Ae^{-2x}(y - 2) \\
 y - yAe^{-2x} &= -2Ae^{-2x} \\
 y &= \frac{2Ae^{-2x}}{Ae^{-2x} - 1}
 \end{aligned}$$

(c) $y' = yx^2;$

$$\begin{aligned}
 \frac{dy}{y} &= x^2 dx \\
 \ln |y| &= \frac{1}{3}x^3 + C \\
 y &= \pm Ae^{\frac{1}{3}x^3}
 \end{aligned}$$

(d) $xy' = \frac{1}{y^3}.$

$$\begin{aligned}
 y^3 dy &= \frac{1}{x} \\
 \frac{y^4}{4} &= \ln |x| + C \\
 y &= (\pm 4 \ln |x| + C)^{1/4}
 \end{aligned}$$

8. Solve the IVPs

(a) $y' = \frac{1}{y} \quad y(0) = 1;$

(b) $y' = -3y + e^{-2x}$ $y(0) = 0$;

(c) $y' = 4y(3 - y)$ $y(0) = 1$;

(d) $y' - \frac{y}{x} = xe^x$ $y(1) = e - 1$;

(e) $y' = 8x^3e^{-2y}$ $y(1) = 0$;

(f) $y' = x^2(1 + y)$ $y(0) = 3$.

Solution:

(a) $y' = \frac{1}{y}$ $y(0) = 1$;

$$y^2 = 2x + C$$

$$C = 1$$

$$y = \sqrt{2x + 1}$$

(b) $y' = -3y + e^{-2x}$ $y(0) = 0$;

$$\rho = e^{3x}$$

$$(ye^{3x})' = e^x$$

$$ye^{3x} = e^x + C$$

$$y = e^{-2x} + Ce^{-3x}$$

$$C = -1$$

$$(c) \quad y' = 4y(3 - y) \quad y(0) = 1;$$

$$\frac{dy}{y(3 - y)} = 4x + C$$

$$\frac{1}{y(3 - y)} = \frac{A}{y} + \frac{B}{(3 - y)}$$

$$1 = A(3 - y) + By$$

$$B = 1/3, A = 1/3$$

$$\frac{1}{3}(\ln |y| - \ln |3 - y|) = 4x + C$$

$$\ln \left| \frac{y}{3 - y} \right| = 12x + C$$

$$\left| \frac{y}{3 - y} \right| = Ae^{12x} \quad A > 0$$

$$\text{since } \frac{y}{3 - y} > 0 \text{ for } y = 1 \implies$$

$$\frac{y}{3 - y} = Ee^{12x}$$

$$y = (3 - y)Ee^{12x}$$

$$y(1 + Ee^{12x}) = 3Ee^{12x}$$

$$y = \frac{3Ee^{12x}}{1 + Ee^{12x}}$$

$$y = \frac{3E}{e^{-12x} + E}$$

$$y = \frac{3}{Fe^{-12x} + 1}, \quad E = 1/F$$

$$y(0) = 1 \implies F = 2 \lim_{x \rightarrow \infty} \frac{3}{2e^{-12x} + 1} = 3$$

$\implies$ stable equilibrium is $y=3$, which makes sense

$$(d) \quad y' - \frac{y}{x} = xe^x \quad y(1) = e - 1;$$

$$\rho = e^{-\ln x} = 1/x$$

$$(y/x)' = e^x$$

$$y/x = e^x + C$$

$$y = xe^x + Cx$$

$$y(0) = e - 1 = e^1 + C \implies C = -1$$

$$(e) \quad y' = 8x^3 e^{-2y} \quad y(1) = 0;$$

$$\int e^{2y} dy = \int 8x^3 dx$$

$$e^{2y}/2 = 2x^4 + C$$

$$y = \ln(4x^4 + C)/2$$

$$y(1) = 0 = \ln(4 + C)/2 \implies C = -3$$

$$(f) \quad y' = x^2(1 + y) \quad y(0) = 3.$$

$$\frac{dy}{y+1} = x^2 dx$$

$$\ln|y+1| = x^3/3 + C$$

$$y = -1 + Ae^{x^3/3}$$

$$A = 4$$

9. **Cooling problem.** A cake is removed from the oven at 210F and left to cool at room temperature, which is 70F. After 30 minutes, the temperature of the cake is 140F. When will it be 100F ? Hint: use Newton's law of cooling.

Solution:

$$\frac{dT}{dt} = k(70 - T) \quad T(0) = 210$$

$$T' + kT = k70$$

$$(e^{kt}T)' = k70e^{kt}$$

$$T(t)e^{kt} = T(0) + 70(e^{kt} - 1)$$

$$T(t) = 210e^{-kt} + 70(1 - e^{-kt})$$

$$T(30) = 140 = 210e^{-k30} + 70(1 - e^{-k30})$$

$$70 = (210 - 70)e^{-k30}$$

$$-k30 = \ln(1/2)$$

$$k = \ln(2)/30 \quad T(t) = 140e^{-kt} + 70 = 100$$

$$e^{-kt} = \frac{30}{140}$$

$$t = \ln\left(\frac{14}{3}\right)/k$$

10. **Mixture problem.** Suppose a 10 liter tank has a salt water input rate of 1/2 liter per minute, with input salt concentration of 2 grams per liter. The output rate is the

same as the input rate, so the tank volume will remain a constant 10 liters for all time. Let $x(t)$ be the amount of salt (in grams) in the tank at time t . Assume at time $t = 0$ minutes that the tank is fresh water ($x(0) = 0$).

- (a) Given the information above, write down a DE for the salt amount $x(t)$.
- (b) Solve the DE using the initial condition $x(0) = 0$.

Solution: $V = 10$, $r_{in} = r_{out} = 1/2$. $c_{in} = 2$. $c_{out} = x(t)/V$.

(a)

$$\begin{aligned}\frac{dx}{dt} &= r(c_{in} - c_{out}) = \frac{1}{2}\left(2 - \frac{x}{10}\right) \\ &= 1 - \frac{1}{20}x \\ x(0) &= 0\end{aligned}$$

(b)

$$\begin{aligned}\frac{dx}{dt} + \frac{x}{20} &= 1 \\ \rho &= e^{t/20} \\ (xe^{t/20})' &= e^{t/20} \\ xe^{t/20} &= 20(e^{t/20} - 1) \\ x(t) &= 20(1 - e^{-t/20})\end{aligned}$$

(c)

$$\lim_{t \rightarrow \infty} x(t)/10 = 2 \text{ [g/L]}$$

Which makes sense given the input concentration is 2 grams per liter, then the tank will be the same in the limit.

11. **Acceleration problem.** Consider a boat which starts at rest at time $t = 0$ sec, is accelerated in a straight path by an engine that provides a constant $1000N$ of force, and which is also subject to drag forces of $40N$ for each m/s of velocity. The mass of the boat is $400kg$.

- (a) Show that the boat velocity $v(t)$ in m/s satisfies the IVP

$$v'(t) = 2.5 - 0.1v \quad v(0) = 0.$$

HINT: Derive the equations from the Newton's 2nd law of physics.

- (b) Solve the IVP.

(c) How far does the boat travel in the first 20 seconds ?

Solution:

$$ma = mv' = F = 1000 - 40v$$

$$m = 400$$

$$v' = 2.5 - \frac{1}{10}v$$

$$v' + \frac{1}{10}v = 2.5$$

$$(e^{\frac{1}{10}t}v)' = 2.5e^{\frac{1}{10}t}$$

$$e^{\frac{1}{10}t}v = 25(e^{\frac{1}{10}t} - 1)$$

$$v(t) = 25(1 - e^{-\frac{1}{10}t})$$

$$x(t) = 25(t + 10[e^{-\frac{1}{10}t} - 1])$$

$$x(20) = 500 + 100[e^{-2} - 1]$$

12. Find the row reduced echelon form of the matrices

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -4 & -2 \\ 3 & -12 & 1 \\ 2 & -8 & 5 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

13. Suppose the matrix equation $\mathbf{Ax} = \mathbf{b}$ is row-equivalent to the reduced augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Find the solution of the linear system.

Solution:

$$x_4 = 1$$

$$x_3 = s$$

$$x_2 = 2 - 3s$$

$$x_1 = 1 - 2s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix},$$
$$s \in \mathbb{R}$$

So, no inverse exists.

14. Determine if the matrix $\mathbf{A}$ has an inverse or not

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Solution:

$$\text{Det}(\mathbf{A}) = 0$$

15. Sketch an accurate picture of the set $\mathbf{Ax}$ if $\mathbf{x}$ is all vectors in the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Solution:

