

PRACTICE FINAL EXAM

- (10 pts) 1. Define an equilibrium point to a differential equation $\frac{dy}{dt} = f(y)$. How would you accurately determine stability of an equilibrium point?

Solution: An equilibrium y^* is a number in which $f(y^*) = 0$. Stability occurs only when $f_y(y^*) < 0$.

- (10 pts) 2. Determine the stability and phase diagram on the entire real line of the DE $\frac{dy}{dt} = -\sin(\pi y)$

Solution: The integers $y^* = n$ are all equilibrium points. $f_y(y^*) = -\pi \cos(n\pi)$ is negative for $n = 0$ and therefore stable. The odd integers are unstable, the even integers are stable.

- (10 pts) 3. Population equation. Suppose a population of 100 bacteria are observed to double in 30 minutes. Assuming exponential growth, write down the DE that describes the bacterial population $P(t)$ every minute.

Solution:

$$dP/dt = kP$$

where $P(t) = 100e^{kt}$. When $t = 30$, then

$$200 = P(30) = 100e^{k30}$$

, so

$$\ln(2)/30 = k$$

- (10 pts) 4. Suppose a yeast bioreactor is known to follow logistic growth, with maximum carrying capacity of $M = 100$ kg of yeast and growth rate constant $k = 1/10$. Let the yeast in kg be $Y(t)$, which follows the DE

$$\frac{dY}{dt} = kY(M - Y).$$

What initial population of yeast $Y(0) = Y_0$ will produce the maximum instantaneous growth rate at $t = 0$?

Solution:

$$dY/dt = (1/10)Y(100 - Y)$$

The maximal growth rate is where dY/dt is largest. Examining the right hand side when $Y = Y_0$, we find its maximum point when its derivative with respect to Y is zero:

$$\partial_y(1/10)Y(100 - Y) = 100 - 2Y = 0$$

Which occurs when $Y = Y_0 = 50$. That is, halfway to the maximum growth.

- (10 pts) 5. Suppose the birth/death rate per month per fish for a population of fish in a lake is $k(t) = \sin\left(\frac{2\pi}{12}(t-3)\right)$, with $P(0) = 100$. Write down an expression for the population after a year ($t = 12$)?

Solution: The DE is:

$$\frac{dP}{dt} = \sin\left(\frac{2\pi}{12}t\right)P$$

Solving

$$\rho(t) = e^{\int \sin\left(\frac{2\pi}{12}(t-3)\right) dt} = e^{-\frac{12}{2\pi} \cos\left(\frac{2\pi}{12}(t-3)\right)}$$

$$(\rho P)' = 0$$

$$\rho P = C$$

$$P(t) = Ce^{\frac{12}{2\pi} \cos\left(\frac{2\pi}{12}(t-3)\right)}$$

$$P(0) = 100 = Ce^{\frac{12}{2\pi} \cos\left(\frac{6\pi}{12}\right)} \implies C = 100e^{-\frac{12}{2\pi} \cos\left(\frac{\pi}{2}\right)} = 100$$

$$P(12) = 100e^{\frac{12}{2\pi} \cos\left(\frac{3\pi}{2}\right)} = 100$$

- (10 pts) 6. Find the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$$\implies \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\text{check: } \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

(10 pts) 7. Are the following vectors linearly independent, or dependent:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}?$$

Justify your answer.

Solution: No. The first vector times -2 produces the second vector.

(10 pts) 8. We define the set S of vectors $\vec{u} = [u_1 \ u_2 \ u_3]^T$ as $2u_1 + u_3 = 1$ and $u_2 = 2$. Is S a subspace or not? Verify the properties of a subspace in order to earn full credit.

Solution: No. To prove, find a vector that satisfies the conditions of set membership: $\vec{u} = [1 \ 2 \ -1]^T$. Now check if a scalar multiple is also a solution: Note that $2\vec{u} = [2 \ 4 \ -2]^T$ which has a second entry that is not 2, so the set S is not a subspace.

(10 pts) 9. We define the set S of vectors $\vec{u} = [u_1 \ u_2 \ u_3]^T$ as $2u_1 + u_2 = 0$ and $u_2 = 0$. Is S a subspace or not? Verify the properties of a subspace in order to earn full credit.

Solution: We must establish that the set is closed under vector addition for any two set members to prove its a subspace (this is the only condition that must be checked): Suppose $\vec{u} = [u_1 \ u_2 \ u_3]^T$ and $\vec{v} = [v_1 \ v_2 \ v_3]^T$ are members of S . The sum

$$\vec{w} = \vec{v} + \vec{u} = [u_1 + v_1 \ u_2 + v_2 \ u_3 + v_3]^T,$$

where

$$2w_1 + w_2 = 2(u_1 + v_1) + u_2 + v_2 = (2u_1 + u_2) + (2v_1 + v_2) = (0) + (0) = 0$$

So the first condition of S holds. Note that because $u_2 = 0$ and $v_2 = 0$ then $w_2 = 0$ as well, so both conditions of S hold, and so the vector sum $\vec{v} + \vec{u}$ is also a member of S . Therefore, S is a subspace.

Another way to prove this is a subspace is to draw a picture of the set in 3D and verify its a line passing through the origin in the x-y plane with slope 2.

(10 pts) 10. Find the homogeneous solution space to the DE:

$$x'' + 5x' + 4x = 0$$

Solution: The characteristic equation:

$$r^2 + 5r + 4 = (r + 1)(r + 4) = 0 \implies r = -1, -4$$

The solution space is spanned by the set $\{e^{-4t}, e^{-t}\}$.

(10 pts) 11. Find the particular solution to the non-homogeneous DE:

$$L(x) = x'' + 4x' + 6x = \sin(2t)$$

Solution: The characteristic equation:

$$r^2 + 4r + 4 + 2 = (r + 2)^2 + 2 = 0 \implies r = -2 \pm i\sqrt{2}$$

The solution space is spanned by the set $\{e^{-2t} \cos(\sqrt{2}t), e^{-2t} \sin(\sqrt{2}t)\}$. Since $\sin(2t)$ is not in the span of the homogeneous solution space, we try our candidate particular solution to be:

$$x_p = A \cos(2t) + B \sin(2t)$$

Putting in the differential operator, and equating with the right hand side $\sin(2t)$, we get

$$x_p' = -2A \sin(2t) + 2B \cos(2t)$$

$$x_p'' = -4A \cos(2t) - 4B \sin(2t)$$

$$L(x_p) = -4A \cos(2t) - 4B \sin(2t) + 4(-2A \sin(2t) + 2B \cos(2t)) + 6(A \cos(2t) + B \sin(2t)) = \sin(2t)$$

$$(-4A + 8B + 6A) \cos(2t) \implies 2A + 8B = 0$$

$$(-4B - 8A + 6B) \sin(2t) = \sin(2t) \implies -8A + 2B = 1$$

$$\implies A = -4B \implies 32B + 2B = 34B = 1 \implies B = 1/34$$

$$A = -4/34 = -2/17$$

(10 pts) 12. Solve using Laplace transforms and the convolution formula for a generic $f(t)$.

$$\frac{dx}{dt} = -2x + f(t)$$

with initial condition $x(0) = 0$.

Solution:

$$X(s)(s + 2) = F(s)$$

$$X(s) = \frac{1}{s + 2} F(s)$$

$$x(t) = \int_0^t e^{-2\tau} f(t - \tau) d\tau$$

(10 pts) 13. Compute the following convolution product:

$$x(t) = (f * g)(t) = \int_0^t g(t - \tau)f(\tau)d\tau$$

where $g(t) = e^{-t}$ and $f(t) = te^{-2t}$.

Solution:

$$\begin{aligned} x(t) &= (f * g)(t) = \int_0^t e^{-(t-\tau)}\tau e^{-2\tau}d\tau \\ &= e^{-t} \int_0^t e^{\tau}\tau e^{-2\tau}d\tau = e^{-t} \int_0^t \tau e^{-\tau}d\tau \\ &= e^{-t}([- \tau e^{-\tau}]_0^t - [e^{-\tau}]_0^t) \\ &= e^{-t}(-te^{-t} + (1 - e^{-t})) \\ &= -te^{-2t} + e^{-t} - e^{-2t} \end{aligned}$$

(10 pts) 14. Solve the second-order system $\mathbf{M}\vec{x}'' = \mathbf{K}\vec{x}$ with

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$$

With initial conditions

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{x}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Recall that solutions are of the form $\vec{x}(t) = \vec{v}_1(a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)) + \vec{v}_2(a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t))$

Solution:

The eigenvalues are $\lambda = -4, -1$ with eigenvectors

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $\omega_2 = \sqrt{4} = 2, \quad \omega_1 = 1$

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (a_1 \cos(2t) + b_1 \sin(2t)) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_2 \cos(t) + b_2 \sin(t))$$

The initial conditions equations:

$$\begin{aligned} \vec{x}(0) &= a_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \vec{x}'(0) &= \begin{bmatrix} -1 \\ 2 \end{bmatrix} (2b_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} (b_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

which are solved simply enough by you, the students.

(10 pts) 15. Solve the IVP with a Laplace transform method

$$x'' + 2x' + 2x = e^{-t}, \quad x(0) = 1, \quad x'(0) = 1$$

Solution: $x(t) = 2e^{-t} \sin(t) + e^{-t}$ Check the text for use of partial fractions techniques used to solve.

16. Solve the linear system and plot the first 2π seconds of the trajectory in the phase plane. Indicate what direction the solution is heading from start to finish. Hint: $e^{-1} \approx 0.37$.

$$\vec{x}' = \mathbf{A} \vec{x}$$

with

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

Recall that solutions involving complex eigenvalues $\lambda = a \pm bi$ and eigenvectors $\vec{a} \pm i \vec{b}$ are of the form $\vec{x}(t) = c_1 e^{at} [\vec{a} \cos(bt) - \vec{b} \sin(bt)] + c_2 e^{at} [\vec{b} \cos(bt) + \vec{a} \sin(bt)]$

Solution: The eigenvalues are $\lambda = -1 \pm i = a \pm ib$. The eigenvectors are

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{a} \pm i \vec{b}$$

The solution is then

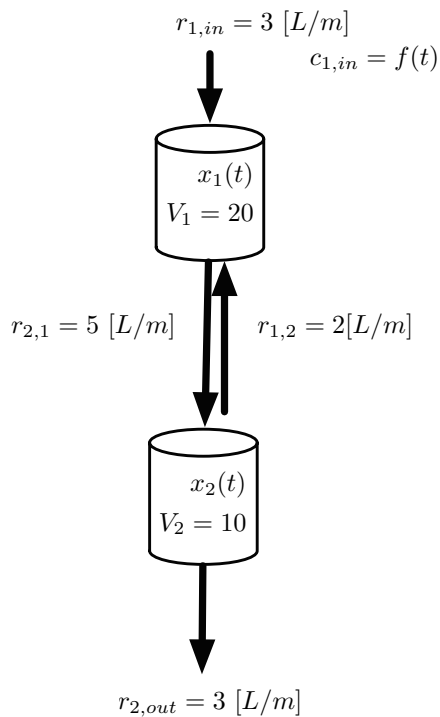
$$\vec{x}(t) = c_1 e^{-t} [(\vec{a} \cos(t) - \vec{b} \sin(t)) + c_2 e^{-t} [\vec{b} \cos(bt) + \vec{a} \sin(bt)]]$$

. The initial condition determines c_1 and c_2 :

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which implies that $c_1 = 1$ and $c_2 = 1$. After 2π seconds, one full revolution will have been completed through a decaying spiral, to the point $[0.37 \quad 0.37]^T$

(10 pts) 17. Write down, but do not solve the linear system described as follows.



Solution:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} -5/20 & 2/10 \\ 5/20 & -5/10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3f(t) \\ 0 \end{bmatrix}$$

(10 pts) 18. Consider the non-linear system

$$\begin{aligned} \frac{dx}{dt} &= -x + xy \\ \frac{dy}{dt} &= x - y + y^2. \end{aligned}$$

Find all equilibrium points and determine stability.

Solution:

$$0 = -x + xy = x(y - 1)$$

so $x = 0$ and $y = 1$ are candidates. Check the second equation:

$$0 = -x - y + y^2 = -x - y(1 - y)$$

so $x = 0$ and $y = 0$ and $y = 1$ are candidates. The intersection of these candidates is $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$

The jacobian is

$$J(x, y) = \begin{bmatrix} -1 + y & x \\ -1 & -1 + 2y \end{bmatrix}$$

At the first equilibrium, the Jacobian is

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

which has eigenvalues $\lambda = -1, -1$, so the origin is stable.

At the second equilibrium, the Jacobian is

$$J(0,1) = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

which has a characteristic equation

$$\lambda(\lambda - 1) = 0$$

so the eigenvalues are $\lambda = 0, 1$. Because one is positive, the point $(0, 1)$ is unstable.