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SINC SOLUTION OF BIHARMONIC PROBLEMS

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ABSTRACT. In this paper we solve two biharmonic problems over a square, $B = (-1,1) \times (-1,1)$. (1) The problem $\nabla^4 U = f$, for which we determine a particular solution, U, given f, via use of Sinc convolution; and (2) The boundary value problem $\nabla^4 V = 0$ for which we determine V given $V = g$ and normal derivative $V_{\mathbf{n}} = h$ on ∂B, the boundary of B. The solution to this problem is carried out based on the identity

$$V = \Re\left\{\overline{(z-c)}\,\mathcal{E} + \mathcal{F}\right\} = (x-a)\,u + (y-b)\,v + \phi\,,$$

where $\mathcal{E} = u + i\,v$ and $\mathcal{F} = \phi + i\,\psi$ are functions analytic in B, and where $c = a + i\,b$ is an arbitrary constant. We thus determine approximations to the harmonic functions u, v and ϕ on ∂B, via use of Sinc quadrature, and Sinc approximation of derivatives. We then use a special, explicit Sinc-based analytic continuation procedure to extend the functions u, v and ϕ to the interior of B. These procedures enable us to determine functions W which solve a boundary problem of the form $\nabla^4 W = f$ in B, given f in B and given W and its normal derivative, $W_{\mathbf{n}}$ on the boundary of B.

Given any $\varepsilon > 0$, the time complexity of sequential computation of an approximation of W_ε to W to within a uniform error of ε in B, i.e., such that $\sup_{(x,y)\in B} |W(x,y) - W_\varepsilon(x,y)| < \varepsilon$, is $O\big((\log(\varepsilon))^6\big)$.

1 Introduction and summary

This paper deals with the numerical solution of the PDE problem,

$$\begin{aligned} &\nabla^4\, W(x,y) = f(x,y), \quad (x,y) \in B \\ &W(x,y) = g(x,y) \quad \text{and} \quad \frac{\partial U(x,y)}{\partial \mathbf{n}} = h(x,y), \quad (x,y) \in \partial B\,. \end{aligned} \tag{1.1}$$

Here B is assumed to be the square, i.e., $B = (-1,1)\times(-1,1) \subset \mathbb{R}^2$ and $\mathbf{n} = \mathbf{n}(x,y)$ denotes the outward normal at (x,y) on ∂B, the boundary of

B. We shall assume, for simplicity, that f, g and h are given, real-valued functions.

The computation of the solution of the above problem consists of three parts.

Part I. *Particular Solution of* $\nabla^4 U = f$. We first construct a particular solution, U—i.e., without resort to satisfying boundary conditions—of the above PDE problem by evaluating of the Green's function integral expression

$$U(x,y) = \int\int_B \mathcal{G}(x-\xi\,, y-\eta)\, f(\xi\,,\eta)\, d\xi\, d\eta\,, \tag{1.2}$$

via use of *Sinc convolution*. This is accomplished by means of the program package [**15**], the theory of which is described in [**13**]. Essential details of this procedure are given in Section 2 which follows.

Part II. *Solution of the BIE*. Here we describe an explicit procedure, based on use of analytic functions, to solve the homogeneous boundary value problem

$$\begin{aligned} &\nabla^4\, V(x,y) = 0, \quad (x,y) \in B \\ &V(x,y) = g(x,y), \quad \text{and} \quad \frac{\partial V(x,y)}{\partial \mathbf{n}} = h(x,y), \quad (x,y) \in \partial B, \end{aligned} \tag{1.3}$$

where g and h are given. We thus describe an explicit procedure, based on Sinc approximation, for obtaining the boundary values functions $\mathcal{E}$ and $\mathcal{F}$ which are analytic functions of $z = x + i\, y$ in the region $B = \{(x,y) : -1 < x < 1\,, \quad -1 < y < 1\}$, and where $c = a + i\, b$ is an arbitrary constant, such that

$$V = \Re\left\{ \overline{(z-c)}\ \mathcal{E} + \mathcal{F} \right\} = (x-a)\, u + (y-b)\, v + \phi\,. \tag{1.4}$$

It may be shown [**8**] that the right hand side of (1.4) satisfies the equation $\nabla^4 V = 0$ and moreover, every solution to $\nabla^4 V = 0$ is of the form (1.4). In the form (1.4) it thus suffices to determine the real parts of $\mathcal{E}$ and $\mathcal{F}$ on the boundary of B (since, with $u = \Re\mathcal{E}$, we also have $v = \Im\mathcal{E} = \mathcal{S}u$, where $\mathcal{S}$ denotes the Hilbert transform, see [**20**]), by the requirement that V must also satisfy the boundary conditions, for $(x,y) \in \partial B$.

Part III. *Analytic Continuation of* V *into* B. This part of the solution can be carried out via a *Sinc procedure* based on §4.3 of [**18**] and §4.3.13

of [**13**]; given a function g defined on ∂B, we are able to construct an approximation to a function u that is harmonic in B, where $u = g$ on ∂B. Indeed, our procedure yields a solution V that is uniformly accurate in B, even when g and h have singularities, or even discontinuities at the corners of B.

Part IV. *Examples.* In Section 4 the above procedures are tested via a Matlab program, using the Sinc package [**15**]. Given functions f, g and h, we solve problem (1.1) according to the following procedure:

(a) First determine a function U via accurate approximation of the integral (1.2) by use of Sinc convolution. This phase is described in greater detail in Section 2 below. We thus get an approximation of the particular solution U given by (1.2) at the Sinc points $(x_j\,, x_k)$ in the interior of B. We also compute approximations of the partial derivatives, U_x and U_y via use of Sinc convolution.

(b) Solve the PDE $\nabla^4\, V = 0$ for V at the Sinc points of ∂B via use of the analytic function procedure described in Section 3, subject to the given boundary conditions

$$\begin{aligned} V(x, y) &= g(x, y), \\ \frac{\partial V(x, y)}{\partial \mathbf{n}} &= h(x, y). \end{aligned} \tag{1.5}$$

We thus determine approximate values of the functions u, v and ϕ at the Sinc points of ∂B.

(c) Determine v on ∂B from values of u on ∂B via Sinc approximation of Hilbert transforms.

(d) Use the analytic continuation procedure of Section 4 to extend the definitions of the harmonic functions u, $v$and ϕ into B, i.e., evaluate these functions at the Sinc points $(x_j\,, x_k)$ in the interior of B, and then form the solution $V = (x - a)\, u + (y - a)\, v + \phi$ at these same Sinc points.

The time complexity of our method, i.e., the amount of time required to construct an approximation W_ε to W, such that $\sup_{(x,y)\in\partial B} |W(x, y) - W_\varepsilon(x, y)| \le \varepsilon$ is $O\big((\log(\varepsilon))^6\big)$.

2 The particular solution We illustrate here the approximate evaluation of the integral (1.2), which satisfies the equation

$$\nabla^4 U = f \quad \text{in} \quad B = (-1, 1) \times (-1, 1). \tag{2.1}$$

Our procedure for this evaluation is via use of Sinc convolution. The theory of one dimensional Sinc convolution and its extension to rectangular regions in more than one dimension may be found in §4.6 of [**18**], and while unnecessary for our purposes in this paper, extensions to multi-dimensional curvilinear regions may be found in §5.5 of [**13**]. Here we give only an algorithmic description of this procedure, i.e., without proof.

2.1 Sinc parameters and matrices Let us define numbers σ_k and e_k, by

$$\begin{aligned} \sigma_k &= \int_0^k \operatorname{sinc}(x)\, dx, \quad k \in \mathbb{Z}, \\ e_k &= \frac{1}{2} + \sigma_k, \end{aligned} \tag{2.2}$$

and let us then define an $m \times m$ Toeplitz matrix I_m by $I_m = [e_{j-k}]$, with e_{j-k} denoting the $(j,k)^{th}$ element of I_m. For $j = -N, \ldots, N$, let $t_j = (e^{jh} - 1)/(e^{jh} + 1)$ denote the Sinc points of the interval $(-1, 1)$, and let $w_j = 2\, h\, e^{jh}/(1+e^{jh})^2$ denote the corresponding Sinc quadrature weights. Let D denote the diagonal matrix, $D = \operatorname{diag}(w_{-N}, \ldots, w_N)$. Set $P = I_m\, D$, and $Q = (I_m)^T\, D$. If $r(t)$ is a function defined on the interval $(-1, 1)$, set $r_j = r(t_j)$, and form the column vector $\mathbf{r} = (r_{-N}, \ldots, r_N)^T$. Then, for each $j = -N, \ldots, N$, the component R_j^+ of the vector $\mathbf{R}^+ = \left(R_{-N}^+, \ldots, R_N^+\right)^T = A\, \mathbf{r}$ is an accurate approximation of the function $R^+(x) = \int_{-1}^x r(t)\, dt$ evaluated at $x = t_j$. Similarly, the component R_j^- of the vector $Q\, \mathbf{r}$ is an accurate approximation to $R^-(x) = \int_x^1 r(t)\, dt$ evaluated at the points $x = t_j$.

We next need to diagonalize the matrices P and Q above, in the form

$$P = X\, S\, Xi, \qquad Q = Y\, S\, Yi, \tag{2.3}$$

where X and Y are the matrices of eigenvectors of P and Q respectively, $Xi = X^{-1}$, $Yi = Y^{-1}$, and where S is a diagonal matrix of eigenvalues s_j, $j = -N, \ldots, N$. We may note that P and Q have the same eigenvalues.

2.2 "Laplace transform" of the Green's function The Sinc convolution procedure for approximating the function U in (1.2) requires the use of the two dimensional "Laplace transform" G of the Green's function $\mathcal{G}$, where

$$\mathcal{G}(x,y) = \frac{1}{8\pi}\left(x^2+y^2\right)\,\log\left(\sqrt{x^2+y^2}\right). \tag{2.4}$$

The "Laplace transform" G of $\mathcal{G}$ is derived in [**13**]. It is given by

$$\begin{aligned} G(s,\sigma) = \widehat{\mathcal{G}}(s,\sigma) &= \int_0^\infty \int_0^\infty \mathcal{G}(x,y)\,\exp\left(-\frac{x}{s}-\frac{y}{\sigma}\right)\,dx\,dy \\ &= \left(\frac{1}{s^2}+\frac{1}{\sigma^2}\right)^{-2}\left(\frac{1}{4}+R(s,\sigma)\right), \end{aligned} \tag{2.5}$$

where

$$R(s,\sigma) = \frac{1}{8\pi}\,\left((7-6\,\gamma)\,\left(\frac{s}{\sigma}+\frac{\sigma}{s}\right)+T(s,\sigma)+T(\sigma,s)\right), \tag{2.6}$$

where γ denotes Euler's constant, and where

$$T(s,\sigma) = \log(s)\,\left(\frac{6s}{\sigma}+\frac{2\,s^3}{\sigma^3}\right). \tag{2.7}$$

Moreover, in order to approximate U_x and U_y in B, we also need the "Laplace transforms" of the derivatives of the Green's function. The partial derivatives of U can be also computed on B and ∂B via use of Sinc convolutions, using the "Laplace transforms"

$$\begin{aligned} \widehat{\mathcal{G}_x}(s,\sigma) &= \frac{1}{s}\,G(s,\sigma) - \frac{\sigma^3}{4\pi}\,(3-2\,\gamma+2\,\log(\sigma))\,, \\ \widehat{\mathcal{G}_y}(s,\sigma) &= \frac{1}{\sigma}\,G(s,\sigma) - \frac{s^3}{4\pi}\,(3-2\,\gamma+2\,\log(s))\,. \end{aligned} \tag{2.8}$$

2.3 Computation of U , U_x , and U_y via Sinc convolution Next, we form two $m \times m$ matrices, $F = [f(t_j,t_k)]$ and $G = [G(s_j,s_k)]$.

We then form an $m \times m$ matrix $\mathbf{U}$ via execution of the following four-line Matlab program, which is just the two dimensional Sinc convolution algorithm for approximating the integral (1.2):

```
U = X*(G.*(Xi*F*Xi.'))*X.';
U = U + Y*(G.*(Yi*F*Xi.'))*X.';
U = U + X*(G.*(Xi*F*Yi.'))*Y.';
U = U + Y*(G.*(Yi*F*Yi.'))*Y.';
```

The resulting matrix $\mathbf{U}$ now has the property that the $(j,k)^{th}$ element of $\mathbf{U}$ is an approximation to the function $U(t_j, t_k)$, where U is defined in (1.2).

The partial derivatives U_x and U_y my be computed via a similarly simple algorithm. Note, however, due to requirement of consistency with the original "models" p and q as given in Eqs. (4.6.1) of [**18**], due to the fact that $\mathcal{G}_x(x,y)$ is an odd function of x, we first write the derivative with respect to x of (1.2) in the form

$$U_x(x,y) = \int_{-1}^{x} \int_{-1}^{1} \mathcal{G}_x(x-\xi\,, y-\eta)\, f(\xi,\eta)\, d\xi\, d\eta - \int_{x}^{1} \int_{-1}^{1} \mathcal{G}_x(\xi - x, y-\eta)\, f(\xi,\eta)\, d\xi\, d\eta\,. \tag{2.9}$$

Thus, if GX denotes the $m \times m$ matrix of the two dimensional "Laplace transform" of $\mathcal{G}_x(x,y)$ evaluated at the eigenvalues (s_j, s_k), the Matlab convolution algorithm for approximating U_x in B takes the form:

```
Ux = X*(GX.*(Xi*F*Xi.'))*X.';
Ux = Ux + Y*(GX.*(Yi*F*Xi.'))*X.';
Ux = Ux - X*(GX.*(Xi*F*Yi.'))*Y.';
Ux = Ux - Y*(GX.*(Yi*F*Yi.'))*Y.';
```

Similarly, if GY denotes the $m \times m$ matrix of the two dimensional "Laplace transform" of $\mathcal{G}_y(x,y)$, the Matlab convolution algorithm for approximating U_y in B takes the form:

```
Uy = X*(GY.*(Xi*F*Xi.'))*X.';
Uy = Uy - Y*(GY.*(Yi*F*Xi.'))*X.';
Uy = Uy + X*(GY.*(Xi*F*Yi.'))*Y.';
Uy = Uy - Y*(GY.*(Yi*F*Yi.'))*Y.';
```

3 Solution of the homogeneous boundary value problem In this section we shall determine boundary values on ∂B of harmonic functions u and v and ϕ, such that the expression

$$V = \Re[\,\overline{(z-c)}\,\mathcal{E} + \mathcal{F}] = (x-a)\,u + (y-b)\,v + \phi, \tag{3.1}$$

in which $\mathcal{E} = u + i\,v$ and $\mathcal{F} = \phi + i\,\psi$, solves the boundary value problem

$$\nabla^4 V = 0 \quad \text{in} \quad B = (-1,1) \times (-1,1). \tag{3.2}$$

subject to given values of $V = g$ and $V_{\mathbf{n}} = h$. where the subscript "$\{\cdot\}_{\mathbf{n}}$" denoted differentiation in the direction of the outward normal. In (3.1), $\mathcal{E}$ and $\mathcal{F}$ are analytic in B, while u, v, ϕ and ψ are harmonic in B. For the sake of simplicity, we assume the constant c has the form $c = (1+i)\,a$, where a is real-valued.

As discussed at length in [**8**], every solution V of this boundary value problem can be represented via use of the above expression (3.1).

It will suffice to just determine u and ϕ on ∂B, since v can be expressed as the Hilbert transform of u over ∂B. Let us describe the explicit details for setting up the system of algebraic equations to determine u and ϕ. After solving for u and ϕ on the boundary, ∂B, we shall also determine v on ∂B, which will then enable us to determine V on the interior of B, via an analytic continuation procedure based on Sinc methods, and which is described in Section 4 below.

3.1 The boundary equations We deduce from (3.1) and (1.5) that

$$\begin{aligned} &(x-a)\,u + (y-a)\,v + \phi = g, \\ &x_{\mathbf{n}}\,u + (x-a)\,u_{\mathbf{n}} + y_{\mathbf{n}}\,v + (y-a)\,v_{\mathbf{n}} + \phi_{\mathbf{n}} = h, \end{aligned} \tag{3.3}$$

where the subscript $\{\cdot\}_{\mathbf{n}}$ denotes differentiation in the direction of the outward normal.

3.2 The oriented boundary of B We parametrize $\Gamma = \partial B = \bigcup_{j=1}^{4} \Gamma_j$ in an oriented fashion in complex variable notation as follows:

$$\begin{aligned} \Gamma_1 &= \left\{ z = z^1(t) = 1 + i\,t, \quad -1 \le t \le 1 \right\}, \\ \Gamma_2 &= \left\{ z = z^2(t) = i - t, \quad -1 \le t \le 1 \right\}, \\ \Gamma_3 &= \left\{ z = z^3(t) = -1 - i\,t, \quad -1 \le t \le 1 \right\}, \\ \Gamma_4 &= \left\{ z = z^4(t) = -i + t, \quad -1 \le t \le 1 \right\}. \end{aligned} \tag{3.4}$$

Next, we define u^j, v^j, ϕ^j and ψ^j, respectively in terms of u, v, ϕ and ψ as follows:

$$\begin{aligned} &\text{If} \quad z = 1 + i\tau \in \Gamma_1, &&\text{then set} \quad u^1(\tau) = u(z), \\ &\text{If} \quad z = -\tau + i \in \Gamma_2, &&\text{then set} \quad u^2(\tau) = u(z), \\ &\text{If} \quad z = -1 - i\tau \in \Gamma_3, &&\text{then set} \quad u^3(\tau) = u(z), \\ &\text{If} \quad z = \tau - i \in \Gamma_4, &&\text{then set} \quad u^4(\tau) = u(z), \end{aligned}$$

and similarly for v^j, ϕ^j, and ψ^j. In this notation, the boundary $\Gamma = \partial B = \bigcup_{j=1}^4 \Gamma_j$ is *oriented* in a counter-clockwise fashion. This means, of course, as τ increases, we traverse from bottom to top along Γ_1, from right to left along Γ_2, from top to bottom along Γ_3, and from left to right along Γ_4.

3.3 The Hilbert transform and its collocation The Hilbert transform relates u and v along Γ as follows, for $z \in \Gamma$:

$$
\begin{aligned}
v(\zeta) = \mathcal{S} u(z) &= \frac{P.V.}{\pi\, i} \int_\Gamma \frac{u(\zeta)}{z-\zeta}\, d\zeta, \\
u(\zeta) = \mathcal{S} v(z) &= \frac{P.V.}{\pi\, i} \int_\Gamma \frac{v(\zeta)}{z-\zeta}\, d\zeta.
\end{aligned} \tag{3.5}
$$

It then readily follows, assuming that u and v are real-valued, if $z \in \Gamma_j$ we must have

$$
\Re \left\{ \frac{P.V.}{\pi\, i} \int_\Gamma \frac{u(\zeta)}{z-\zeta}\, d\zeta \right\} = 0. \tag{3.6}
$$

This property enables us to accurately approximate the Hilbert transforms using only Sinc quadrature, i.e., without use of Sinc-Hilbert transform methods (see §1.7 of [**13**] and §5.6 of [**14**]).

For example, if $z = 1 + i\,\tau \in \Gamma_1$, then we have

$$
\begin{aligned}
v^1(\tau) = \frac{1}{\pi} \int_{-1}^{1} \Bigg\{ & \frac{(1-\tau)\, u^2(t)}{(1+t)^2 + (1-\tau)^2} \\
& + \frac{2\, u^3(t)}{4 + (t+\tau)^2} + \frac{(1+\tau)\, u^4(t)}{(1-t)^2 + (1+\tau)^2} \Bigg\} dt.
\end{aligned} \tag{3.7}
$$

That is, we use the Sinc quadrature approximation,

$$
\int_{-1}^{1} p(t)\, dt = \sum_{k=-N}^{N} w_k\, p(t_k), \tag{3.8}
$$

in which t_j are the *Sinc point* $t_k = (e^{kh} - 1)/(e^{kh} + 1)$, while the w_k are the *Sinc weights*, given by $w_k = 2\, h\, e^{kh}/(e^{kh} + 1)^2$. We set $u_j^\ell = u^\ell(t_j)$, we define vectors, $\mathbf{u}^\ell = \left(u_{-N}^\ell, \ldots, u_N^\ell\right)^T$, and similarly for v_j^ℓ and $\mathbf{v}^\ell$, and we define matrices, $P^2 = \left[P^2_{jk}\right]$, $P^3 = \left[P^3_{jk}\right]$, and $P^4 = \left[P^4_{jk}\right]$ by means of the equations

$$\begin{aligned}
\left[P^2_{jk}\right] &= \frac{(1-t_j)\, w_k}{\pi\left((1-t_j))^2 + (1+t_k)^2\right)}\,, \\
\left[P^3_{jk}\right] &= \frac{2\, w_k}{\pi\left(4 + (t_j + t_k)^2\right)}\,, \\
\left[P^4_{jk}\right] &= \frac{(1+t_j)\, w_k}{\pi\left((1+t_j)^2 + (1-t_k)^2\right)}\,.
\end{aligned} \tag{3.9}$$

Thus, we get $\mathbf{v}^1 = P^2\,\mathbf{u}^2 + P^3\,\mathbf{u}^3 + P^4\,\mathbf{u}^4$. Because of our above definition of $\Gamma = \partial B = \bigcup_{j=1}^4 \Gamma_j$ as an oriented arc, and because of the symmetry of B, we similarly obtain $\mathbf{v}^2 = P^2\,\mathbf{u}^3 + P^3\,\mathbf{u}^4 + P^4\,\mathbf{u}^1$, and so on, i.e., we get the following block system of equations:

$$\begin{bmatrix} \mathbf{v}^1 \\ \mathbf{v}^2 \\ \mathbf{v}^3 \\ \mathbf{v}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & P^2 & P^3 & P^4 \\ P^4 & \mathbf{0} & P^2 & P^3 \\ P^3 & P^4 & \mathbf{0} & P^2 \\ P^2 & P^3 & P^4 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \mathbf{u}^3 \\ \mathbf{u}^4 \end{bmatrix}. \tag{3.10}$$

These equations will be used below, initially to eliminate v, and later, to compute the function v on $\Gamma = \partial B$.

3.4 The equations on the oriented boundary The first equation in (3.3) for $z = x + i\,y$ reduces to the following four equations on the arcs Γ_j, for $j = 1, 2, 3, 4$:

$$\begin{aligned}
(1-a)\, u^1(t) + (t-a)\, \mathcal{S}\, u^1(t) + \phi^1(t) &= g^1(t), \\
(-t-a)\, u^2(t) + (1-a)\, \mathcal{S}\, u^2(t) + \phi^2(t) &= g^2(t), \\
(-1-a)\, u^3(t) + (-t-a)\, \mathcal{S}\, u^3(t) + \phi^3(t) &= g^3(t), \\
(t-a)\, u^4(t) + (-1-a)\, \mathcal{S}\, u^4(t) + \phi^4(t) &= g^4(t).
\end{aligned} \tag{3.11}$$

Let us now turn to the second equation in (3.3). The Cauchy-Riemann equations enable us to write this equation in the form

$$y_{\mathbf{t}}\, u + (x-a)\, v_{\mathbf{t}} - x_{\mathbf{t}}\, v - (y-a)\, u_{\mathbf{t}} + \psi_{\mathbf{t}} = h, \tag{3.12}$$

where the subscript $\{\cdot\}_{\mathbf{t}}$ denotes differentiation along Γ. Now using the notation of (3.4), and noting, e.g., that $(t-a)\, u_t = \{(t-a)\, u\}_t - u$, we can rewrite this equation along each arc Γ_j in the form

$$
\begin{aligned}
2\,u^1 - \{(t-a)\,u^1\}_t + (1-a)\,v_t^1 + \psi_t^1 &= h^1, \\
2\,v_2 - \{(t+a)\,v^2\}_t - (1-a)\,u_t^2 + \psi_t^2 &= h^2, \\
-2\,u^3 + \{(t+a)\,u^3\}_t - (1+a)\,v_t^3 + \psi_t^3 &= h^3, \\
-2\,v^3 + \{(t-a)\,v^4\}_t + (1+a)\,u_t^4 + \psi_t^4 &= h^4.
\end{aligned}
\tag{3.13}
$$

3.5 Collocation of the equations Some *basic* matrices we shall use are $\mathbf{0}$, the $m \times m$ matrix of zeros, $\mathbf{I}$, the identity matrix, D, the Sinc-derivative matrix, which replaces a vector $\mathbf{p} = (p(z_{-N}), \ldots, p(z_N))^T$ by an approximation to $\mathbf{q} = (q(z_{-N}), \ldots, q(z_N))^T$, where $q(z) = p'(z)$, i.e., $D\mathbf{p} = \mathbf{q}$, the diagonal matrix $T = \operatorname{diag}(t_{-N}, \ldots, t_N)^T$ of Sinc points, as well as the representation (3.10), which will enable us to eliminate the function v in the above equations.

By using these matrices we are able to reduce the solution of the system of eight equations (3.11) and (3.13) into an (eight) $\times$ (eight) block system of the form

$$
\mathbf{B}\,\mathbf{w} = \mathbf{k}
\tag{3.14}
$$

where

$$
\mathbf{B} = \begin{bmatrix}
B^{11} & B^{12} & B^{13} & B^{14} & B^{15} & B^{16} & B^{17} & B^{18} \\
B^{21} & B^{22} & B^{23} & B^{24} & B^{25} & B^{26} & B^{27} & B^{28} \\
B^{31} & B^{32} & B^{33} & B^{34} & B^{35} & B^{36} & B^{37} & B^{38} \\
B^{41} & B^{42} & B^{43} & B^{44} & B^{45} & B^{46} & B^{47} & B^{48} \\
B^{51} & B^{52} & B^{53} & B^{54} & B^{55} & B^{56} & B^{57} & B^{58} \\
B^{61} & B^{62} & B^{63} & B^{64} & B^{65} & B^{66} & B^{67} & B^{68} \\
B^{71} & B^{72} & B^{73} & B^{74} & B^{75} & B^{76} & B^{77} & B^{78} \\
B^{81} & B^{82} & B^{83} & B^{84} & B^{85} & B^{86} & B^{87} & B^{88}
\end{bmatrix},
\tag{3.15}
$$

$$
\mathbf{w} = \begin{bmatrix} \mathbf{u}^1 \\ \mathbf{u}^2 \\ \mathbf{u}^3 \\ \mathbf{u}^4 \\ \Phi^1 \\ \Phi^2 \\ \Phi^3 \\ \Phi^4 \end{bmatrix}, \qquad
\mathbf{k} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \\ \mathbf{g}^4 \\ \mathbf{h}^1 \\ \mathbf{h}^2 \\ \mathbf{h}^3 \\ \mathbf{h}^4 \end{bmatrix}.
\tag{3.16}
$$

Here, e.g., $\Phi^1 = \left(\phi^1_{-N}, \ldots, \phi^1_N\right)^T$, etc.

Also, using the convenient definitions $Zm = Z - a * \mathbf{I}$, $Zp = Z + a * \mathbf{I}$, $Wm = 2 * \mathbf{I} - D * Zm$, and $Wp = 2 * \mathbf{I} - D * Zp$, we can now make the following definitions for the entries of the above matrix $\mathbf{B}$. To this end, we omit $m \times m$ zero matrices. First, for the collocation of (3.11) we shall use the nonzero matrices:

$$\begin{aligned}
&B^{11} = (1-a)\,\mathbf{I}, && B^{12} = Zm\,P^2, && B^{13} = Zm\,P^3,\\
&B^{14} = Zm\,P^4, && B^{15} = \mathbf{I}, && \\
&B^{21} = (1-a)\,P^4, && B^{22} = -Zp, && B^{23} = (1-a)\,P^2,\\
&B^{24} = (1-a)\,P^3, && B^{26} = \mathbf{I}, && \\
&B^{31} = -Zp\,P3, && B^{32} = -Zp\,P^4, && B^{33} = -(1+a)\,\mathbf{I},\\
&B^{34} = -Zp\,P^2, && B^{37} = \mathbf{I}, && \\
&B^{41} = -(1+a)\,P^2, && B^{42} = -(1+a)\,P^3, && B^{43} = -(1+a)\,P^4,\\
&B^{44} = Zm, && B^{48} = \mathbf{I}. &&
\end{aligned} \tag{3.17}$$

Next, the nonzero matrices for the collocation of (3.13) are:

$$\begin{aligned}
&B^{51} = Wm, && B^{52} = (1-a)\,DP^2, && \\
&B^{53} = (1-a)\,DP^3, && B^{54} = (1-a)\,DP^4, && \\
&B^{56} = DP^2, && B^{57} = DP^3, && B^{58} = DP^4,\\
&B^{61} = WpP^4, && B^{62} = -(1-a)\,D, && \\
&B^{63} = WpP^2, && B^{64} = WpP^3, && \\
&B^{65} = DP^4, && B^{67} = DP^2, && B^{68} = DP^3,\\
&B^{71} = -(1+a)\,DP^3, && B^{72} = -(1+a)\,DP^4, && \\
&B^{73} = -Wp, && B^{74} = -(1+a)DP^2, && \\
&B^{75} = DP^3, && B^{76} = DP^4, && B^{78} = DP^2,\\
&B^{81} = -WmP^2, && B^{82} = -WmP^3, && \\
&B^{83} = -WmP^4, && B^{84} = (1+a)D, && \\
&B^{85} = DP^2, && B^{86} = DP^3, && B^{87} = DP^4.
\end{aligned} \tag{3.18}$$

4 Sinc approximation and analytic continuation In this section we describe a procedure for extending the functions u, v and ϕ which were computed on ∂B to all of B.

4.1 Sinc basis It is imperative to first recall the Sinc basis.

The function $\varphi(x) = \log((1+x)/(1-x))$ transforms the interval $(-1,1)$ onto the real line $\mathbb{R}$. The Sinc points are defined corresponding to a spacing h on $\mathbb{R}$ by $t_k = \phi^{-1}(kh) = (e^{kh}-1)/(e^{kh}+1)$. We also set $\rho = \exp(\varphi)$, so that $\rho(x) = (1+x)/(1-x)$. Corresponding to a positive integer N, one usually selects $h = c/N^{1/2}$, and one can then perform Sinc approximation on $(-1,1)$ via the formula

$$f(x) \approx \sum_{k=-N}^{N} f(t_k)\,\omega_k(x). \tag{4.1}$$

The basis functions ω_k are defined in terms of the Sinc function $S(k,h)$, which is given by

$$S(k,h)(u) = \frac{\sin\left(\frac{\pi}{h}(u-kh)\right)}{\frac{\pi}{h}(u-kh)}\,. \tag{4.2}$$

Set

$$\begin{aligned}
h &= \frac{c}{N^{1/2}},\\
\gamma_j &= S(j,h)\circ\varphi, \quad j=-N,\ldots,N,\\
\omega_j &= \gamma_j, \quad j=-N+1,\ldots,N-1,\\
\omega_{-N} &= \frac{1}{1+\rho} - \sum_{j=-N+1}^{N} \frac{1}{1+e^{jh}}\,\gamma_j,\\
\omega_N &= \frac{\rho}{1+\rho} - \sum_{j=-N}^{N-1} \frac{e^{jh}}{1+e^{jh}}\,\gamma_j,\\
\varepsilon_N &= N^{1/2} e^{-(\pi\alpha d N)^{1/2}}.
\end{aligned} \tag{4.3}$$

Suppose, for example, that f is analytic and bounded in the eye-shaped region $D = \{z \in \mathbb{C} : |\arg(\varphi(z))| < d\}$, where d is a positive constant, and corresponding to numbers $\alpha \in (0,1)$ and $C > 0$, we have, for $x \in (-1,1)$, the inequality,

$$\left.\begin{array}{lll} |f(x) - f(-1| & \text{if} & x \le 0 \\ |f(x) - f(1)| & \text{if} & x > 0 \end{array}\right\} \le C\,(1-x^2)^{\alpha}.$$

Then, by selecting $h = (\pi d/(\alpha N))^{1/2}$, we obtain a uniform bound on the error in the above Sinc approximation of f is of the order of $N^{1/2}\exp(-(\pi d\alpha N)^{1/2})$.

4.2 Analytic continuation Given g defined and continuous on a closed, finite line segment $\Gamma_\ell = [a_1, a_2]$ in the complex plane, we can define $\varphi^\ell(z) = \log((z-a_1)/(a_2-z))$, and in the notation of the above subsection, we set $Lg(z) = (g(a_1)+\rho(z)g(a_2))/(1+\rho(z)$, which reduces to the linear interpolant,

$$L\,g(x) = \frac{(a_2-z)\,g(a_1)+(z-a_1)\,g(a_2)}{a_2-a_1}. \tag{4.4}$$

The line segment Γ_ℓ may be parametrized in terms of a real variable t by the equation

$$z = z(t) = \frac{a_1+a_2}{2} + \frac{a_2-a_1}{2}\,t. \tag{4.5}$$

We mention that this linear expression has an obvious extension to complex t.

Next, we introduce the following *harmonic basis*, which is defined in the upper complex plane, $\{\Im\,\varphi(z) > 0\}$:

$$\begin{aligned}
\sigma_k(z) &= \Im\left\{\frac{e^{i\pi[\varphi(z)-kh]/h}-1}{\pi[\varphi(z)-kh]/h}\right\}, \quad k \in \mathbb{Z},\\
s_1(z) &= \left[1-\frac{\Im\varphi(z)}{\pi}\right]\Re\left\{\frac{1}{1+\rho(z)}\right\} - \frac{\Re\varphi(z)}{\pi}\Im\left\{\frac{1}{1+\rho(z)}\right\},\\
s_2(z) &= \left[1-\frac{\Im\varphi(z)}{\pi}\right]\Re\left\{\frac{\rho(z)}{1+\rho(z)}\right\} - \frac{\Re\varphi(z)}{\pi}\Im\left\{\frac{\rho(z)}{1+\rho(z)}\right\},\\
\theta_k(z) &= \sigma_k(z), \quad -N < k < N,\\
\theta_{-N}(z) &= s_1(z) - \sum_{n=-N+1}^{N}\frac{1}{1+e^{nh}}\sigma_n(z),\\
\theta_N(z) &= s_2(z) - \sum_{n=-N}^{N-1}\frac{e^{nh}}{1+e^{nh}}\sigma_n(z).
\end{aligned} \tag{4.6}$$

As already implied above, the functions θ_j are harmonic in the region $D^+ \equiv \{\Im(z/(a_2-a_1)) > 0\}$. Suppose now, that we parametrize both

$z = z(t)$ and $\zeta = \zeta(\tau)$ by the above equation, where $z(t) \in \Gamma_\ell$, and where $\tau = t + i\,r$ with $r > 0$. Then $\zeta(\tau) \in D^+$, and it follows moreover, that

$$\lim_{r \to 0} \theta_j(\zeta(t + ir)) = \omega_j(t), \tag{4.7}$$

where ω_j is defined as in (4.3) above. We may note in particular, that if $t = t_k$, then the right hand side of (4.7) reduces to $\delta_{j,k}$, the Kronecker delta.

Suppose, now that we are given a function u on $\Gamma = \partial B$, with $B = (-1, 1) \times (-1, 1)$. We can define a sequence of $2\,N + 1$ functions $\left\{\theta_j^\ell\right\}$ for each of the four arcs Γ_ℓ, defined as in (3.4), thus emabling us to analytically continue u into the interior of B by means of the approximation

$$u(z) \approx u_N(z) = \sum_{\ell=1}^{4} \sum_{k=-N}^{N} c_k^\ell\, \theta_k^\ell(z), \tag{4.8}$$

which is harmonic in B. We can then define vectors $\mathbf{g}^\ell = \left(g_{-N}^\ell, \ldots,\ g_N^\ell\right)^T$ where, with Γ_ℓ parametrized by (3.4), we take $g_j^\ell = g(z^\ell(t_j))$, with t_j the Sinc point, $(e^{jh} - 1)/(e^{jh} + 1)$. Then, by similarly defining $\mathbf{u}^\ell$, we use collocation based on the evaluation of the right hand side of (3.44) at the Sinc points of Γ_j, to arrive at the following system of equations for the vectors $\mathbf{c}^\ell = \left(c_{-N}^\ell, \ldots, c_N^\ell\right)^T$:

$$\begin{bmatrix} \mathbf{I} & K^2 & K^3 & K^4 \\ K^4 & \mathbf{I} & K^2 & K^3 \\ K^3 & K^4 & \mathbf{I} & K^2 \\ K^2 & K^3 & K^4 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{c}^1 \\ \mathbf{c}^2 \\ \mathbf{c}^3 \\ \mathbf{c}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \\ \mathbf{g}^4 \end{bmatrix}. \tag{4.9}$$

Here, the matrices K^ℓ are defined as follows: If we substitute $z = z^1(t_j)$ into (4.8) above, we get

$$c_j^1 + \sum_{k=-N}^{N} \left[\theta_k^2(z^1(t_j))\, c_k^2 + \theta_k^3(z^1(t_j))\, c_k^3 + \theta_k^4(z^1(t_j))\, c_k^4\right] = g^1(z^1(t_j)).$$

The set of these equations for $j = -N, \ldots, N$ may be written in the matrix form

$$\mathbf{I}\mathbf{c}^1 + K^2\mathbf{c}^2 + K^3\mathbf{c}^3 + K^4\mathbf{c}^4 = \mathbf{g}^1,$$

with $K^\ell = \left[\theta_k^\ell(z^1(\tau_j)\right]$, $\ell = 2, 3, 4$. Next, similarly taking a point $z^\ell(t_j)$ on Γ_ℓ, we find that no new matrices of the above type K^q arise, due to the symmetry, and because of our definition of an oriented Γ in (3.4).

We remark that this procedure produces a solution that is uniformly accurate in B, even though the function u may have discontinuities at the corner points of ∂B.

Once we have solved (4.9) for the constants c_j^ℓ, we can use (4.8) to compute u at the Sinc points (t_j, t_k) in the interior of B. We can proceed similarly for v, and then for ϕ, to get U as defined in (3.1) at the Sinc points in B.

5 Rates of convergence of approximations We briefly discuss here the convergence and rate of convergence of the procedures in the previous sections of this paper.

1) Evaluation of the Particular Solution. We discuss here the evaluation of the integral (1.2) via use of Sinc convolution.

As discussed in [**18,** §4.6], Sinc convolution is actually a Cartesian product operation, which can be carried out one dimension at a time. The convergence of these processes was already discussed in [**18,** §4.6]. Exponential convergence, i.e., with error of the order of $\exp(-cN^{1/2})$ is guaranteed, in essence, if: With $\varphi(z) = \log((1+z)/(1-z))$, and α an number between 0 and 1, $f(\cdot, y)/(\varphi'(\cdot))^2$ is analytic on $(-1, 1)$ and belongs to $\mathbf{Lip}_\alpha[-1, 1]$ for each fixed $x \in [-1, 1]$; and dually, $f(x, \cdot)/(\varphi'(\cdot))^2$ is analytic on $(-1, 1)$ and belongs to $\mathbf{Lip}_\alpha[-1, 1]$ for each fixed $y \in [-1, 1]$.

2) Solution of the Homogeneous Boundary Value Problem. The boundary value problem stated as in (3.3) is one discussed at length in [**8**]. Sinc methods for solution of such problems are discussed in [**18,** §6.7], and an extension of this procedure is given in [**1**]. Exponential convergence with error of the order of $\exp(-c\,N^{1/2})$ is guaranteed if the functions g^j and h^j/φ' are analytic on $(-1, 1)$ and of class $\mathbf{Lip}_\alpha[-1, 1]$, for $j = 1, 2, 3, 4$, and if the condition number of the matrix B in (3.16) of the order of N^c for some finite c. To this end, we expect that the condition number of the matrix B will be bounded for almost all choices of the parameter a in our method of solution. Although we have not been able to prove this, our *a posteriori* calculations show that this is indeed the case.

3) Analytic Continuation. Under our above assumed properties on the functions f, g and h, it follows that the functions u^j, v^j, and ϕ^j

are analytic and of class $\mathbf{Lip}_\alpha[-1,1]$. The computation of v in terms of u via use of (3.10) was carried out via use of Sinc quadrature. Hence the maximum error in the computation of v via use of (3.10) is of the order of $\exp(-c\,N^{1/2})$. Our *a posteriori* calculations show that the matrix in (4.9) which was obtained via use of (4.6) and (4.8) has a condition number bounded by 2. In essence, then, we are performing Sinc interpolation to get the constants c_j^ℓ, i.e., we have an error of the order of $\exp(-cN^{1/2})$ since we have assumed that the functions u^j, v^j and ϕ^j are analytic and of class $\mathbf{Lip}_\alpha[-1,1]$. Since the functions θ_k^ℓ in (4.8) are harmonic in B, it thus follows, that the the harmonic extension of the functions u, v and ϕ to the interior of B also has an error of the order of $\exp(-cN^{1/2})$, by the maximum principle for harmonic functions.

6 Numerical examples We now test the above procedures on some specific problems.

6.1 A particular solution in B to the non-homogeneous equation

$$\nabla^4 U = f \quad \text{in} \quad B = (-1,1)\times(-1,1). \tag{6.1}$$

where, with $r = \sqrt{x^2+y^2}$,

$$f(r) = \frac{256}{81}\,\frac{r^4-6r^2+18}{(2-r^2)^{8/3}}\,. \tag{6.2}$$

Such an f is obtained, for example, if we take

$$U = (2-r^2)^{4/3}. \tag{6.3}$$

A plot of the solution to th integral (1.2), i.e.,

$$U(x,y) = \int_{-1}^{1}\int_{-1}^{1} \mathcal{G}(x-\xi, y-\eta)\,(2-\xi^2-\eta^2)^{4/3}\,d\xi\,d\eta\,, \tag{6.4}$$

is given in Figure 1. This integral was evaluated via use of repeated 21-point Sinc quadrature over each of the four rectangles, $(-1,z_j)\times(-1,z_k)$, $(-1,z_j)\times(z_k,1)$, $(z_j,1)\times(-1,z_k)$, and $(z_j,1)\times(z_k,1)$, thus generating a 21^2-point solution at the Sinc points of $B=(-1,1)\times(-1,1)$. On the other hand, Figure 2 illustrate the evaluation of this same integral via use of Sinc convolution, taking 21 points in each variable. This latter approach is of course much more efficient.

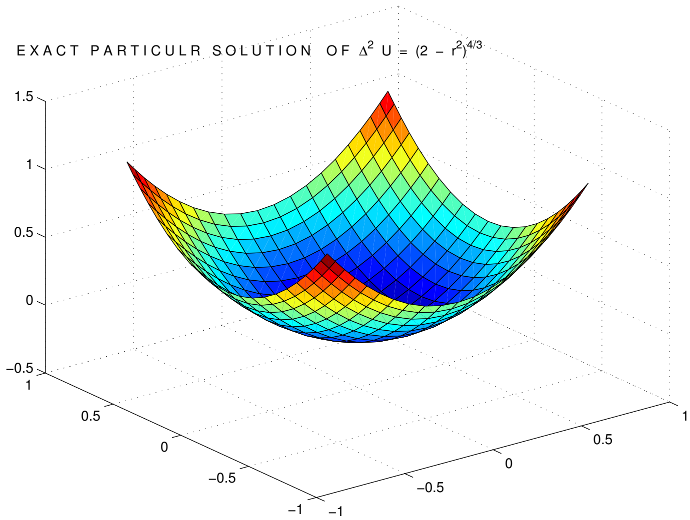

FIGURE 1: "Exact" Evaluation of the Integral (6.4).

6.2 Boundary values of harmonic functions As discussed above, every solution to the equation

$$\nabla^4 V = 0 \quad \text{in} \quad B = (-1,1) \times (-1,1). \tag{6.5}$$

can be expressed in the form

$$V = \Re[\overline{(z-c)}\mathcal{E} + \mathcal{F}] \tag{6.6}$$

with $c = a + ib$ where $\mathcal{E} = u + iv$, $\mathcal{F} = \phi + i\psi$, where u, v, ϕ and ψ are harmonic functions in B, and where a and b are real-valued constants. Specifically, we take $b = a$ and

$$\mathcal{E} = (1 + i - z)^{1/2} \quad \text{and} \quad F = z^4. \tag{6.7}$$

We can deduce that

$$\begin{aligned} u &= ((R + 1 - x)/2)^{1/2}, \\ v &= ((R - 1 + x)/2)^{1/2}, \\ \phi &= x^4 + y^4 - 6\,x^2\,y^2, \\ \psi &= 6\left(x^3\,y + x\,y^3\right), \end{aligned} \tag{6.8}$$

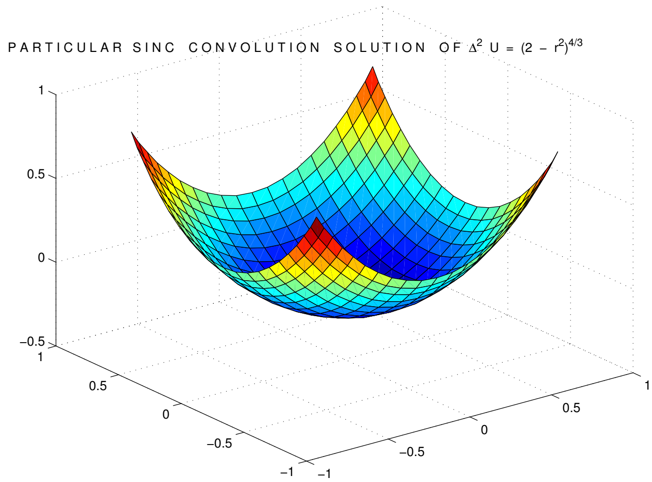

FIGURE 2: "Sinc Convolution" Evaluation of the Integral (6.4).

where $R = ((1-x)^2 + (1-y)^2)^{1/2}$. It then follows that

$$\begin{aligned} u_x &= v_y = -u/(2R), & u_y &= -v_x = v/(2R), \\ \phi_x &= \psi_y = 4x^3 - 12xy^2, & \phi_y &= -psi_x = 4y^3 - 12x^2y. \end{aligned}$$

Notice that each of the functions u and v have a singularity at the point $(1,1)$. Thus, we initially determine the boundary values of $g = V|_{\partial B}$ and $h = (V_{\mathbf{n}})\,|_{\partial B}$, based on the equations (3.3), we eliminate v (and also ψ) based on (3.5) and (3.10), and we then solve the pair of equations

$$\begin{aligned} V &= (x-a)u + (y-a)\mathcal{S}u + \phi = g, \\ \frac{\partial V}{\partial n} &= y_{\mathbf{t}}u + (x-a)(\mathcal{S}\,u)_{\mathbf{t}} - x_{\mathbf{t}}\mathcal{S}v - (y-a)u_{\mathbf{t}} + (\mathcal{S}\phi)_{\mathbf{t}} = h, \end{aligned} \tag{6.9}$$

for u and ϕ on ∂B, (with the subscript $\mathbf{t}$ denoting differentiation along the oriented boundary of B) via solution of the system (3.14). We then again use (3.10) to get v at 21 Sinc points of each arc of ∂B. Finally, we recompute $g_{appr} = V$ on ∂B based on our thus computed u, v and ϕ, and we plot both g_{appr} and the exact values, $g_e x$ at the Sinc points of the four boundary arcs, in Figures 3, 4, 5 and 6.

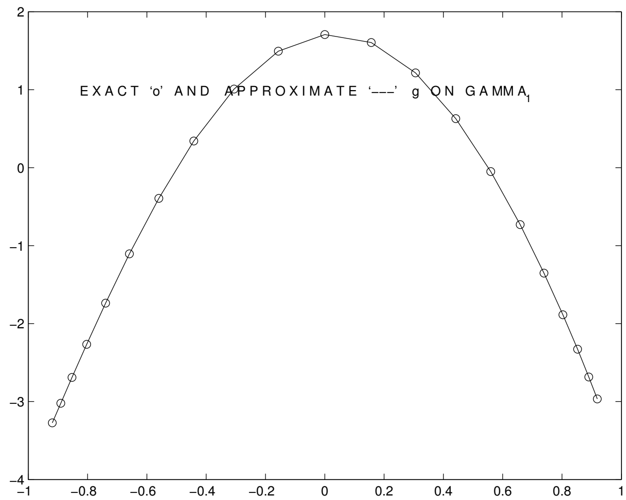

FIGURE 3: "Exact" 'o' and Approximate '–' Values of g on Γ_1.

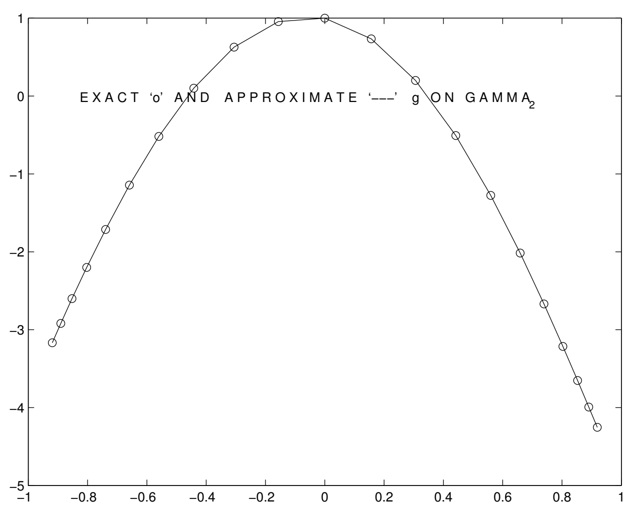

FIGURE 4: "Exact" 'o' and Approximate '–' Values of g on Γ_2.

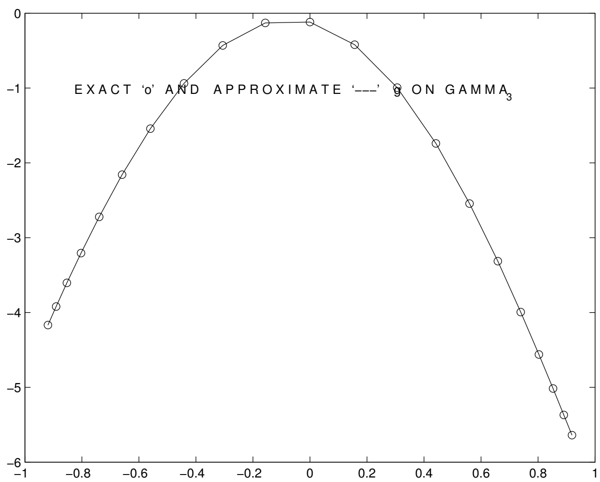

FIGURE 5: "Exact" 'o' and Approximate '–' Values of g on Γ_3.

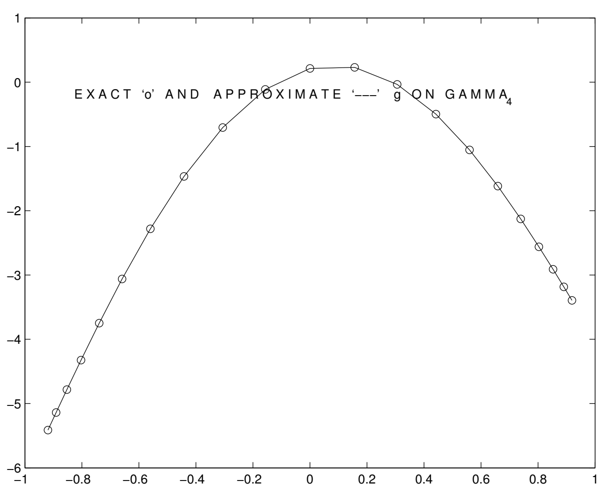

FIGURE 6: "Exact" 'o' and Approximate '–' Values of g on Γ_4.

6.3 Analytic continuation to B of u on ∂B Finally, we use the procedure of Section 4 to analytically continue a function u given on ∂B to the Sinc points in the interior of B. We illustrate respectively, surface plots of u_{ex} in Figure 7, which is the exact harmonic function, as well as u_{appr} in Figure 8, which is the approximate solution computed in this fashion, and the error $u_{appr} - u_{ex}$ in Figure 9.

We remark, that by the maximum principle for harmonic functions, the moduli of the errors of our computed u in B are bounded by the errors on the boundary of B. Of course in performing our computations, we also get round-off errors.

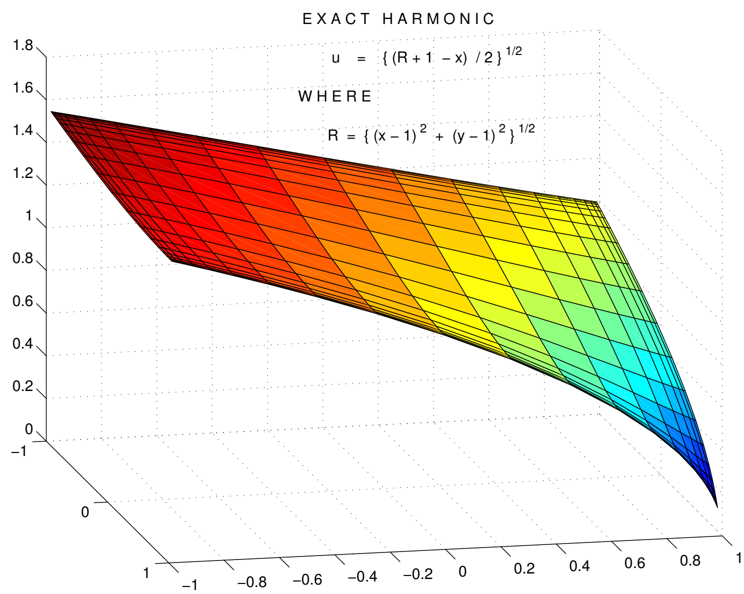

FIGURE 7: Exact Harmonic u in B.

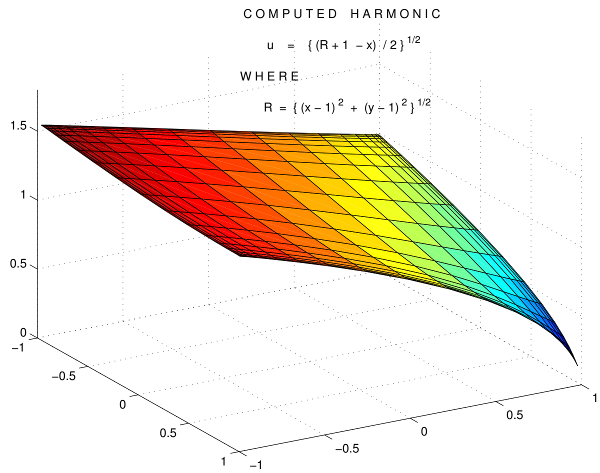

FIGURE 8: Computed Harmonic u in B.

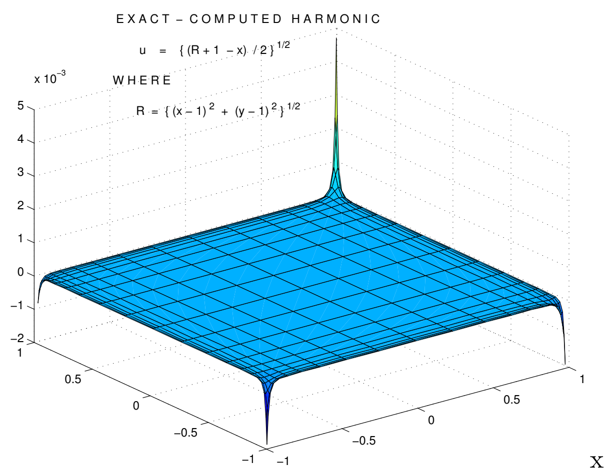

FIGURE 9: $u_{exact} - u_{approx}$ in B.

REFERENCES

1. B. Bialecki, *Sinc-Nyström method for numerical solution of a dominant system of Cauchy singular integral equations given on a piecewise smooth contour*, Siam J. Numer. Anal. **26** (1989), 1194–1211.
2. C T. C. Cook, *Comparison of Finite Difference, Spectral and Sinc Biharmonic Operators*, M.S. thesis, University of Utah, 2004.
3. P. J. Davis, *Orthogonalization Codes in Numerical Analysis*, Ch. 10, *Survey of Numerical Analysis* (J. Todd, ed.), McGraw-Hill, 1962.
4. L. W. Ehrlich, *Solving the biharmonic equation on a square, a direct versus a semidirect method*, Numerical Mathematics **16** (1991), 711–714, 760–788.
5. M. E.-Gamel and J. R. Canon, *Sinc Galerkin method for solving fourth order parabolic differential equations*, to appear in Appl. Numer. Math.
6. M. E.-Gamel, J.R. Canon and A. Zayed, *Sinc Galerkin method for solving sixth order boundary value problems*, to appear.
7. D. Elliott and F. Stenger, *Sinc Method of Solution of Singular Integral Equations*, IMACS Conference on CSIE, Philadelphia, PA (1984), 155–166.
8. F. D. Gakhov, *Boundary Value Problems*, translated from Russian by I. N. Sneddon, Pergamon Press, Oxford, 1966.
9. M. Kowalski, K. Sikorski and F. Stenger, *Selected Topics in Approximation and Computation*, Oxford, 1995.
10. R. Kress, *Linear Integral Equations*, Springer-Verlag, 1989.
11. K. Parker, *Automatic Setup of Sinc-Collocation for Partial differential Equations*, Ph.D. Dissertation, University of Utah, 1999.
12. C. Ralph and K. Bowers, *The Sinc-Galerkin method for fourth order differential equations*, SIAM J. Numer. Anal. **28** (1991), 760–788.
13. $\mathcal{SINC}$, $\mathcal{LLC}$ and F. Stenger, *Tutorial of Sinc Methods*, Part of a completed (in 12-1-03) work of USAF SBIR Contract # F33615–00–C–5004. 300 + iv pages, available in '.pdf'.
14. $\mathcal{SINC}$, $\mathcal{LLC}$ and F. Stenger, *Sinc-Pack Program Manual*, Part of a completed (in 12-1-03) work of USAF SBIR Contract # F33615–00–C–5004. 190 + xi pages, available in '.pdf'.
15. $\mathcal{SINC}$, $\mathcal{LLC}$ and F. Stenger, *SINC-PACK—a Family of Matlab programs*, Part of a completed (in 12-1-03) work of USAF SBIR Contract # F33615–00–C–5004. Over 300 Matlab, programs, based on the theory [**13**] and illustrated in the manual [**14**].
16. R. C. Smith, G. A. Bogar, K. L. Bowers and J. Lund, *The Sinc-Galerkin method for fourth order differential equations*, SIAM J. Numer. Anal. **23** (1991), 760–788.
17. N. Spencer, *Finite Difference, Finite Element, Spectral, and Sinc Numerical Comparisons for a Biharmonic Problem*, M.S. thesis, University of Utah, 2003.
18. F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York, 1993.
19. F. Stenger, *Sinc approximation of Cauchy-type integrals over arcs*, Proceedings of a meeting in Honor of David Elliott's 65^{th} Birthday, 1998.
20. F. Stenger and R. Schmidtlein, *Conformal Maps via Sinc Methods*, Computational Methods in Function Theory (1997) (N. Papamichael, St. Ruscheweyh and E. B. Saff, eds.), World Scientific Publishing Co. Pte. Ltd., (1999), 505–549.
21. Lloyd N. Trefethen, *Spectral Methods in Matlab*, SIAM, Philidelphia, P.A., 2000.

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