L14: Orthogonal Matching Pursuit & Lasso and Compressed Sensing

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Input

$X \in \mathbb{R}^{n \times d}$ $\quad y \in \mathbb{R}^n$

data point $\quad (x_i, y_i) \in (X, y)$

$x_i \in \mathbb{R}^d$

$M_\alpha(x_i) \approx y_i$

$\langle x_i, \alpha \rangle \approx y_i$

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$

Predict: $M(x)$ $\quad x \in \mathbb{R}^d$

where x is new given

y is not

Lasso

$$L_{1,s}(X, y, \alpha) = \sum_{x_i \in X} \| \langle x_i, \alpha \rangle - y_i \|_2^2 + s\|\alpha\|_1$$

$$= \| X\alpha - y \|_2^2$$

How to find α?

Hint: α sparse

most $\alpha_i = 0$

Greedy solving for α_j

$y \approx X\alpha$, $X_{j'}$, $\alpha_{j'}$

- Orthogonal Matching Pursuit (OMP)
- Forward Subset Selection $\|X\alpha - y\|_2^2 + s\|\alpha\|$

Signal $r_{\text{esidual}} = y$; j ; $\alpha = (0, 0, \ldots, 0)$

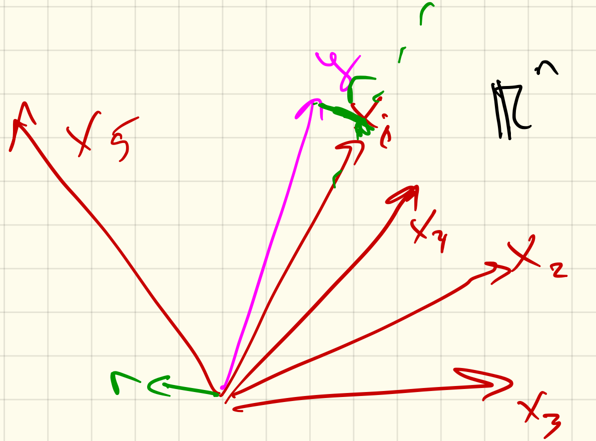
for $i = 1$ to k

$$j_i^* = \operatorname*{arg\,max}_{j' \in [d]} |\langle r, X_{j'} \rangle|$$

$$\alpha_{j_i^*} = \operatorname*{arg\,min}_{\gamma} \|r - X_{j_i^*}\gamma\|^2 + s|\gamma|$$

$$r = r - X_{j_i^*}\alpha_{j_i^*}$$

Return α



Orthogonal Matching Pursuit (OMP)

Find $\alpha^* = \arg\min_{\alpha \in \mathbb{R}^d} \|X\alpha - y\|_2 + s\|\alpha\|_1$
Forward Subset Selection:

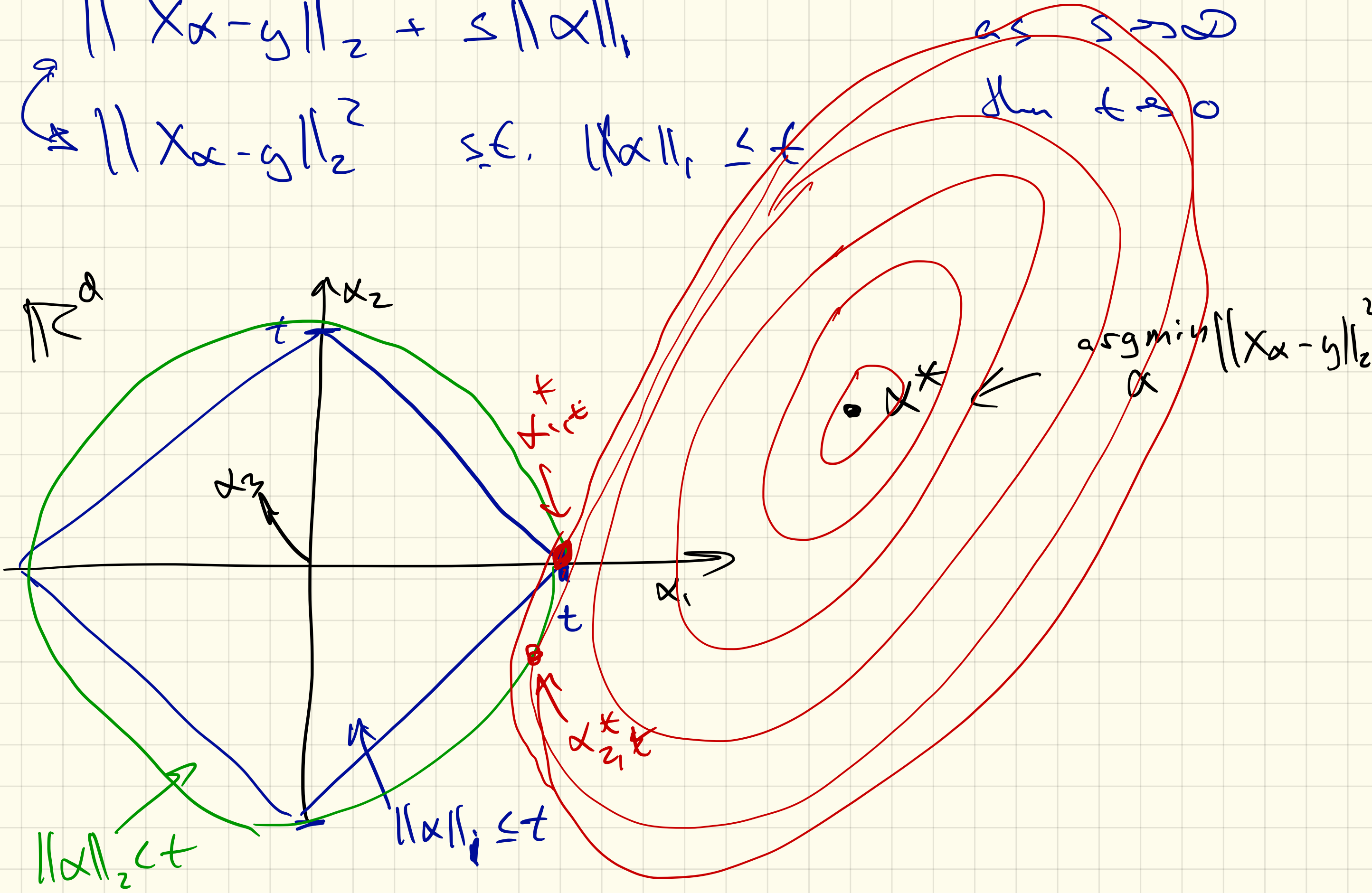
<u>Orthogonal Matching Pursuit</u>

Set $r = y$; $\alpha = \mathbf{0}$.
for $i = 1$ **to** t **do**
  Set $X_j = \arg\max_{X_{j'} \in X} |\langle r, X_{j'} \rangle|$.
  Set $\alpha_j = \arg\min_\alpha \|r - X_j\alpha\| + s|\alpha|$.
  Set $r = r - X_j\alpha_j$.
Return α.

$$\|X\alpha - y\|_2^2 + s\|\alpha\|_1$$

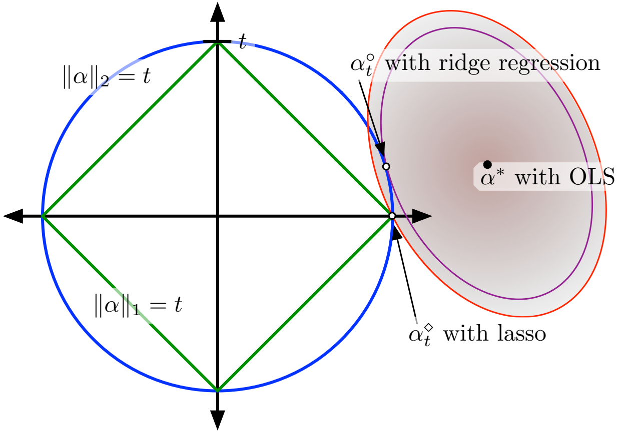
$$\Leftrightarrow \|X\alpha - y\|_2^2 \quad \text{s.t. } \|\alpha\|_1 \le t$$

as $s \to \infty$ then $t \to 0$



Lasso Illustration

Find $\alpha^* = \arg\min_{\alpha \in \mathbb{R}^d} \|X\alpha - y\|_2 + s\|\alpha\|_1$



Compressed Sensing

$$y \approx X \alpha \qquad \alpha^* = \underbrace{(X^T X)}_{d \times d}{}^{-1} X^T y$$

n $\quad$ d

$$d \gg n$$

Lasso: α has $\ll n$ non-zeros.

Sparse Sensing

$m = \#$ nonzeros

$d =$ length of S

$k = \underbrace{C}_{4 \ldots 20} \, m \log\left(\frac{d}{m}\right)$

unknown signal

$S^T = [0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,1\,1\,0\,1\,0\,0\,0\,1\,0\,0\,1\,0\,0]$

$x_i^T = [\text{-}1\,0\,1\,0\,1\,1\,\text{-}1\,1\,0\,\text{-}1\,0\,0\,1\,\text{-}1\,\text{-}1\,1\,0\,1\,0\,1\,\text{-}1\,\text{-}1\,\text{-}1\,0\,1\,0\,0\,\text{-}1\,0\,1\,0\,0]$ sensing vector $\in \{-1, 0, +1\}^d$

observable

$$
\begin{aligned}
y_i &= \langle S, x_i \rangle \\
&= 0{+}0{+}0{+}0{+}1{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}0{+}1{+}0{+}0{+}0{+}1{-}1{+}0{-}1{+}0{+}0{+}0{+}0{+}0{+}0{+}1{+}0{+}0 \\
&= 2
\end{aligned}
$$

Repeat k times

known

observed

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_{12} \end{bmatrix} \quad \cdot \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{12} \end{bmatrix} \begin{bmatrix} S \end{bmatrix}$$

$S = \alpha$

Orthogonal Matching Pursuit (OMP)

Orthogonal Matching Pursuit

Set $r = y$.
for $i = 1$ **to** t **do**
  Set $X_j = \arg\max_{X_{j'} \in X} |\langle r, X_{j'} \rangle|$.
  Set $\gamma_j = \arg\min_{\gamma} \|r - X_j \gamma\|$. ← 1
  Set $r = r - X_j \gamma_j$.
Return $\hat{S}$ where $\hat{s}_j = \cancel{\gamma}_j$ (or 0).

OMP Example

$m = 3$, $d = 10$

$k \approx 20 \cdot m \cdot \log_2\left(\frac{10}{3}\right)$

signal: $S = [0, 0, 1, 0, 0, 1, 0, 0, 1, 0]$

measurement:
$$X = \begin{bmatrix} 0 & 1 & 1 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 \end{bmatrix}$$

observation: $y = XS^T = [0, 0, 0, 1, 1, -2]^T$

0	0	+1	+2	+3	+1	+2	?	+4	0
0	+1	0	0	0	0	0	−1		0

$r_1 = y - X_9 = [1, 0, 0, 0, 0, -1]$

$r_2 = r_1 - X_2 = [0, 1, 1, 0, 0, -1]$

$r_3 = r_2 - X_3 = [-1, 1, 0, 1, 0, 0]$

$r_4 = r_3 + X_2 = [0, 0, -1, 1, 0, 0]$

$r_5 = r_4 - X_6 = [0, 0, 0, 0, 0, 0]$

Lasso Illustration

Least Angles Regression

Find $\alpha^* = \arg\min_{\alpha \in \mathbb{R}^d} \|X\alpha - y\|_2 + s\|\alpha\|_1$