L22: Markov Chains

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April 8, 2019

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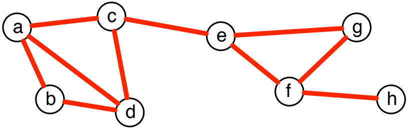
Markov Chain : Life Lessons

- **[L1]** *Only your current position matters going forward, don't worry about the past.*
- **[L2]** *You just need to worry about one step at a time; you will get there eventually (or you won't).*
- **[L3]** *In the limit, everyone has perfect karma.*

Graphs



Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$$E = \Big\{\{a,b\},\{a,c\},\{a,d\},\{b,d\},\{c,d\},\{c,e\},\{e,f\},\{e,g\},\{f,g\},\{f,h\}\Big\}.$$

Matrix-Style: As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

A_3

$G =$

	a	b	c	d	e	f	g	h
a	0	1	1	1	0	0	0	0
b	1	0	0	1	0	0	0	0
c	1	0	0	1	1	0	0	0
d	1	1	1	0	0	0	0	0
e	0	0	1	0	0	1	1	0
f	0	0	0	0	1	0	1	1
g	0	0	0	0	1	1	0	0
h	0	0	0	0	0	1	0	0

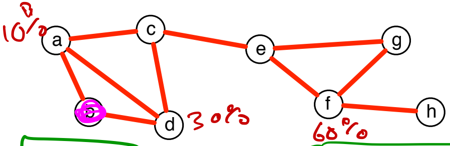
$A =$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Markov Chain

$q_i \geq 0$

$\sum_{i=1}^{n} q_i = 1$



edges

(V, P, q): V node set, P probability transition matrix, q initial state.

e.g. $q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$ or $q^T = [\underbrace{0.1}_{a}\ 0\ 0\ \underbrace{0.3}_{d}\ 0\ \underbrace{0.6}_{f}\ 0\ 0]$. $\in \Delta_n$

1 @ b

ith row

$P_i = \dfrac{A_i}{\|A_i\|_1}$

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix}$$

$P_{ij} \geq 0$

$\sum_{j=1}^{n} P_{ij} = 1$

~~row~~ column normalized

Transitions in (a)

$$
P = \begin{pmatrix}
0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\
1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0
\end{pmatrix} \quad \text{and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]
$$

@ b

states it might go to

$$
q_1 = Pq = \left[\frac{1}{2} \quad 0 \quad 0 \quad \frac{1}{2} \quad 0 \quad 0 \quad 0 \quad 0\right]^T .
$$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

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$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

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In the limit: $q_n = P^n q$

as $n \to \infty$ (need conditions)

Transitions

Markov Chains

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

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In the limit: $q_n = P^n q$

[**L1**] *Only your current position matters going forward, don't worry about the past.*

Two Perspectives of MCs

- random walk

  each state of is at exactly 1 location.

- probability distribution on random walk

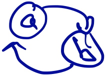
  $q_i \in \Delta_n$

  keeps track of probability of random walk.

Cyclic Examples

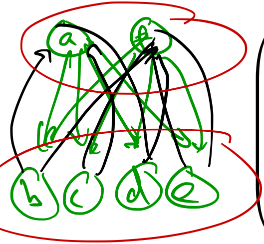
"in the limit"

need "ergodic"



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

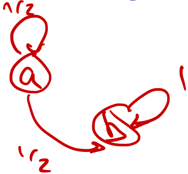
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

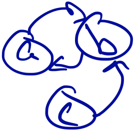


$$\begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \end{pmatrix}$$

bipartite graph

Absorbing and Transient Examples





$$\begin{pmatrix} 1/2 & 0 \\ 1/2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

a, b

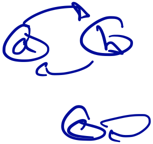
Transient

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 49/100 & 0 & 0 & 0 & 0 \\ 0 & 1/100 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

absorbing

Unconnected Examples

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ergodic

- not cyclic
- no transient
- connected

Limiting State

Assume ergodic

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$. ← in general, not uniform

not $q_* = [\frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n}]$

Limiting State

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.

> [**L2**] *You just need to worry about one step at a time; you will get there eventually (or you won't).*

if not ergodic

Delicate Balance

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.
Also $q_* = PP^* q$ thus $q_* = Pq_*$.

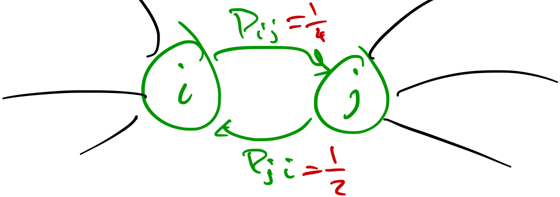
implies

if q close to q_*, needs fewer steps.

So the probability of (being in a state i and leaving to j) is the same as (being in another state j and arriving in i)

$$P_{i,j} q_*(i) = P_{j,i} q_*(j)$$

$$\frac{q_*(i)}{q_*(j)} = \frac{P_{ji}}{P_{ij}}$$



Delicate Balance

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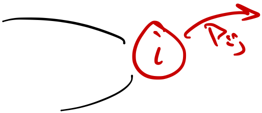
So the probability of (being in a state i and leaving to j) is the same as (being in another state j and arriving in i)

$$P_{i,j} q_*(i) = P_{j,i} q_*(j)$$

[**L3**] *In the limit, everyone has perfect karma.*



Calculate q_*

- $q_* =$ for $i = 1$ to T (power method)

  $q = P q$ mat-vec

- $q_* = P^T = \underbrace{P \cdot P}_{P^2} \cdot \cdots \cdot P$

  $P^4 = (P^2) \cdot (P^2)$

  $P^8 = (P^4)(P^4)$

  $\log T$ matrix mult.

- $q_* =$ Run T random walks

  $\to$ take average state.

- $q_* =$ first eigenvector of P.

  $[V, L] = \text{eig}(P)$

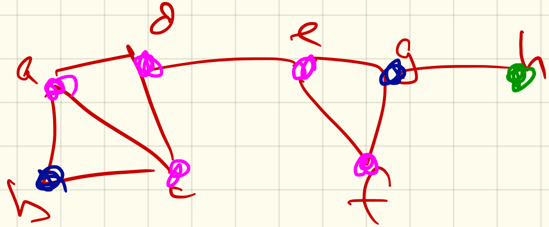
  $v_1 = V(:, 1);$

  $q_* = v_1 / \text{sum}(v_1)$

  $L(2,2) = \lambda_2$

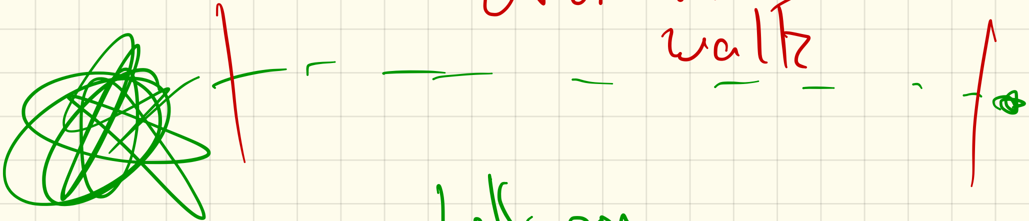
  smaller = faster converg.

Example graph



$$\xi_x = \begin{array}{cccccccc} a & b & c & d & e & f & g & h \\ (0.15, & 0.1, & 0.15, & 0.15, & 0.15, & 0.15, & 0.1, & 0.05) \end{array}$$

drunkard's walk



lollypop

Metropolis Algorithm

Gibbs Sampling

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Metropolis Algorithm

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Likelihood $\approx \frac{\text{Prob}}{Z}$ / $e^{-(\text{energy})}$

Metropolis on V and w

init

Initialize $v_0 = [0\ 0\ 0\ \ldots\ 1\ \ldots\ 0\ 0]^T$.

repeat

  Generate $u \sim K(v, \cdot)$ guess new local state u

  if $(w(u) \geq w(v_i))$ **then**

    Set $v_{i+1} = u$

  else

    With probability $w(u)/w(v)$ set $v_{i+1} = u$

  else

    Set $v_{i+1} = v_i$

v u

until "converged"

return $V = \{v_1, v_2, \ldots, \}$ $\approx w$