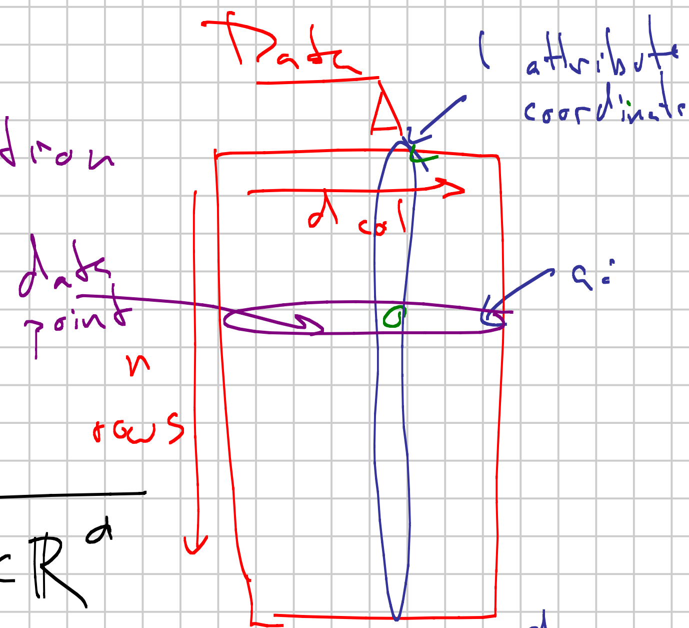
Singular Value Decomposition

Note Title 3/2/2016

Review of SVD

↪ geometric intuition

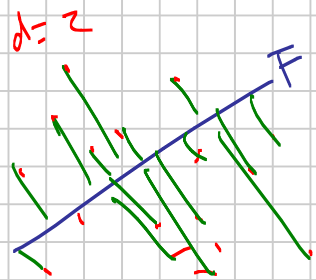
- PCA
- Eigen-decomposition
- MDS



Find mapping $\mu : \mathbb{R}^d \to F \subset \mathbb{R}^d$

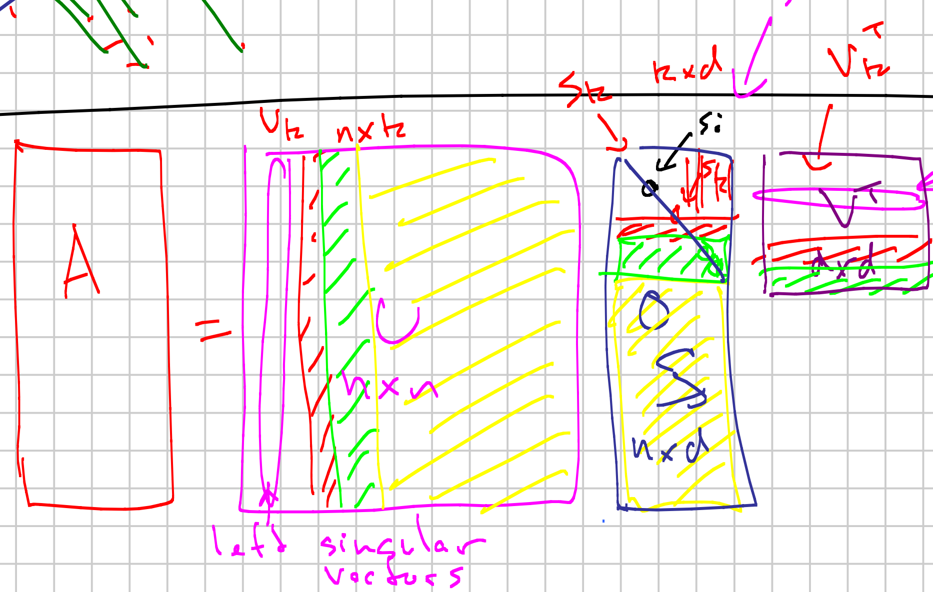
minimize $\sum_i (a_i - \mu(a_i))^2$, $a_i \in A$

$a_i \in A$ $\quad$ $a_i \in \mathbb{R}^d$



F is k-dimensional subspace

singular values



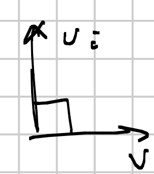
right singular vectors

$A = USV^T$

$[U, S, V] = svd(A)$

U, V orthogonal matrices

- each column $\|u_i\| = 1$
- pair col $u_i, u_j \to \langle u_i, u_j \rangle = 0$
- $V^T = V^{-1} \to V^T V = I$



$S = \mathrm{diag}(s_1, s_2, \dots, s_k, \dots, s_r, 0, 0 \dots) \quad r \le d \equiv \text{rank}$

$s_k \to 0$, $k+1$

$s_i \ge s_{i+1}$

$$\underset{n \times d}{A_k} = \underset{n \times k}{U_k}\ \underset{k \times k}{S_k}\ \underset{k \times d}{V_k^T}$$

$\bar{a}_i$

$$\sum_{a_i \in A} (\mu(a_i) - a)^2 = \|A - A_k\|_F^2$$

$$\|M\|_F^2 = \sum_i M_{ij}^2$$

$$\mu(a_i) = \bar{a}_i$$

$$\|A - A_k\|_2 = s_{k+1}$$

$$\|M\|_2 = \max_{\|x\|=1} \|Mx\|$$

$$\forall a_i \in F \subset \mathbb{R}^d$$

F k-dimensional

$$V = [v_1, v_2, \dots, v_d]$$

$$\alpha_i = \langle x, v_i \rangle$$

for any $x \in \mathbb{R}^d$

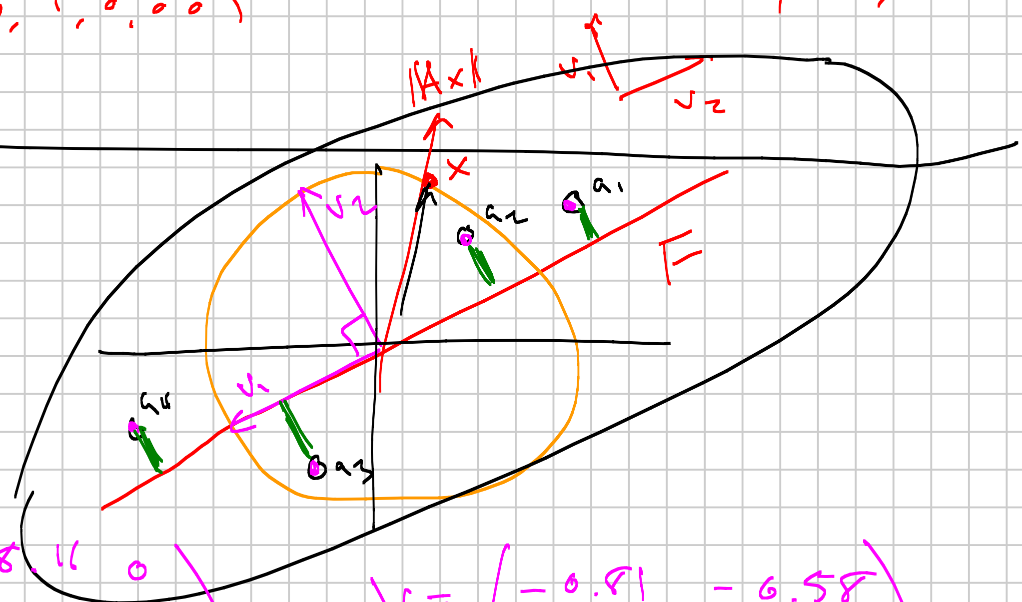
$$x = \sum_{i=1}^{d} \alpha_i v_i$$

orig coord sys — ith entry

$$v_i = e_i = (0, 0, 0, \dots, 1, 0, 0, 0)$$

$$x = (\alpha_1, \alpha_2, \dots, \alpha_d)$$

$$A = \begin{pmatrix} 4 & 3 \\ 2 & 2 \\ -1 & -3 \\ -5 & -2 \end{pmatrix}$$



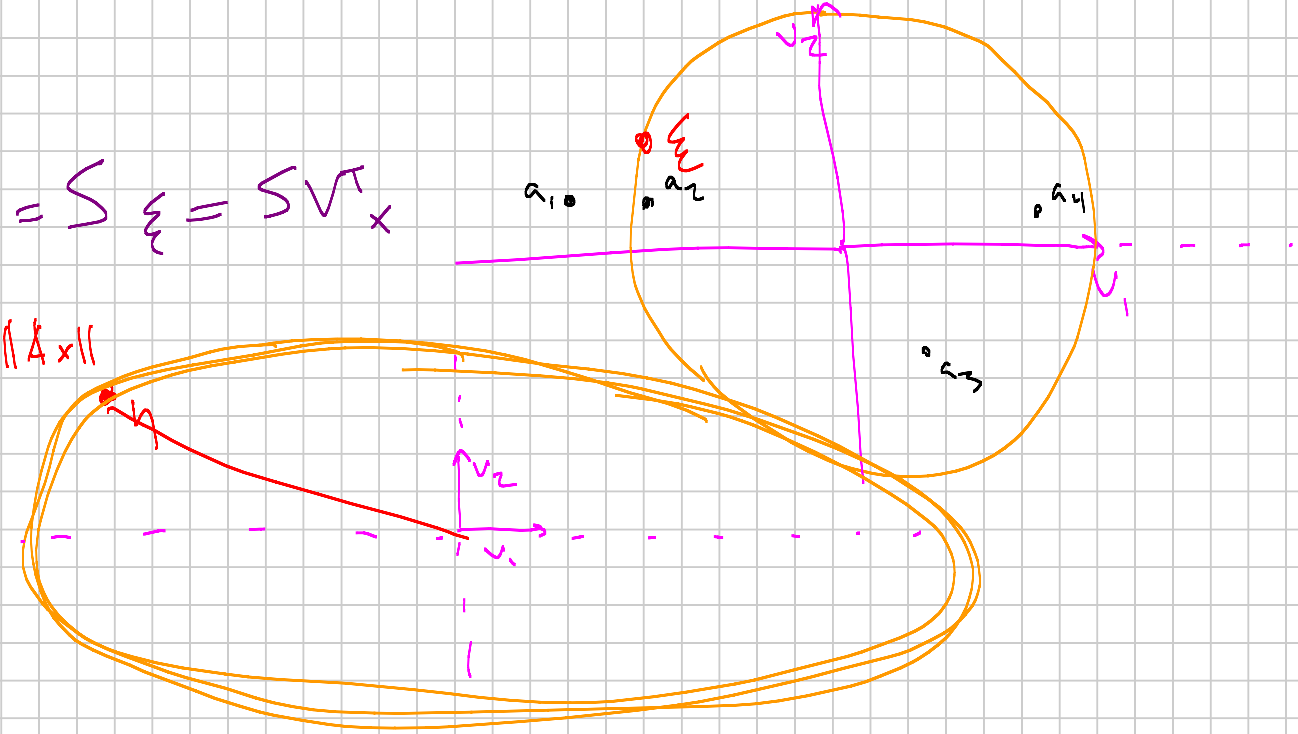
$$S = \begin{pmatrix} 8.11 & 0 \\ 0 & 2.3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} -0.81 & -0.58 \\ -0.58 & 0.81 \end{pmatrix}$$

$$Ax = \underbrace{US\underbrace{V^T x}}$$

$$\xi = V^T x$$

$$\eta = S\,\xi = S V^T x$$

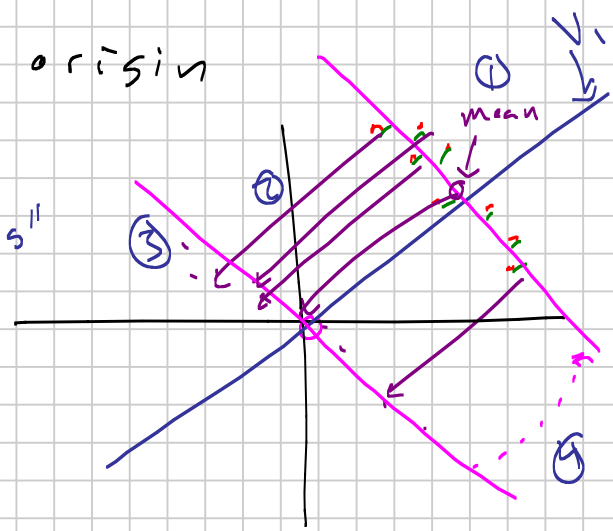
$\|\eta\| = \|Ax\|$



PCA (Principle Component Analysis)

SVD minimized $\|A - A_k\|$
such subspace $F = V_k$ includes
the origin

1. Find mean $m = \text{mean}(A)$
2. For all $a_i \to \tilde{a}_i = a_i - m$ "centering"
3. Run $\text{svd}(\tilde{A}) \Longrightarrow F$
4. Shift $F \to F + m$



$$C_n = I_n - \tfrac{1}{n} \mathbb{1}\mathbb{1}^T \qquad \tilde{A} = C_n A$$

Eigen Decomposition

eigen vector

λ eigenvalue

$$M v = \lambda v$$

square matrix

scalar

$$\underset{d \times d}{A^T A} V = (U S^T \cancel{U^T})(\cancel{U} S \cancel{V^T}) \cancel{V} = V S^2$$

right singular vectors v_i = eigenvector $A^T A$

$s_i^2 = \lambda_i$

$$\underset{n \times n}{A A^T} U = U S^2$$

MDS

Input: distance matrix $D_{n \times n}$

$$D_{ij} = d(p_i, p_j)$$
$$= \| p_i - p_j \|$$

$$A A^T = M$$

"classic MDS"

$$M_{ij} = \langle a_i, a_j \rangle$$

$$\| a_i - a_j \|^2 = \| a_i \|^2 + \| a_j \|^2 - 2 \langle a_i, a_j \rangle$$

$$\| a_k \| = 0 \qquad \| a_j \|^2 = \| a_j - a_1 \|^2$$

$$\langle a_i, a_j \rangle = \frac{1}{2} \left(\| a_i - a_1 \|^2 + \| a_j - a_1 \|^2 - \| a_i - a_j \|^2 \right)$$

$\text{eig}(A A^T) \rightarrow U$ take top k $\Rightarrow \mathbb{R}^k$