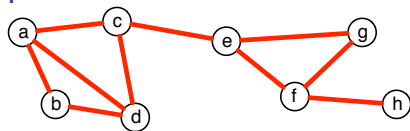


# L10: Spectral Clustering

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# Graphs



**Mathematically:**  $G = (V, E)$  where

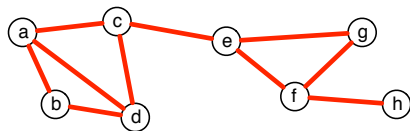
$V = \{a, b, c, d, e, f, g\}$  and

$E = \left\{ \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\} \right\}$ .

**Matrix-Style:** As a matrix with 1 if there is an edge, and 0 otherwise.  
(For a directed graph, it may not be symmetric).

$$G = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline a & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ f & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ g & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

# Laplacian Matrix



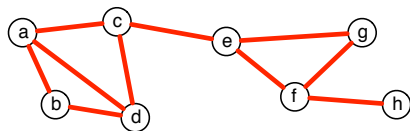
adjacency

diagonal

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

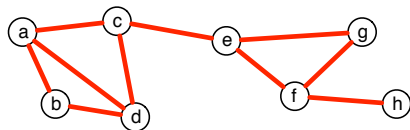
$$D = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Unnormalized Laplacian Matrix



$$L_0 = D - A = \begin{pmatrix} 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

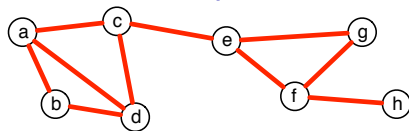
# Unnormalized Laplacian Matrix



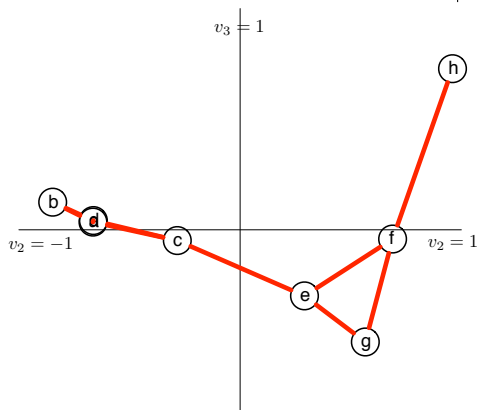
eigenvectors of  $L_0$

$\lambda$	0	<b>0.278</b>	1.11	2.31	3.46	4	4.82
V	$1/\sqrt{8}$	-0.36	0.08	0.10	0.28	0.25	$1/\sqrt{2}$
	$1/\sqrt{8}$	-0.42	0.18	0.64	-0.38	0.25	0
	$1/\sqrt{8}$	-0.20	-0.11	0.61	0.03	-0.25	0
	$1/\sqrt{8}$	-0.36	0.08	0.10	0.28	0.25	$-1/\sqrt{2}$
	$1/\sqrt{8}$	0.17	-0.37	0.21	-0.54	-0.25	0
	$1/\sqrt{8}$	0.36	-0.08	-0.10	-0.28	0.75	0
	$1/\sqrt{8}$	0.31	-0.51	-0.36	-0.56	0.56	0
	$1/\sqrt{8}$	0.50	0.73	0.08	0.11	0.11	0

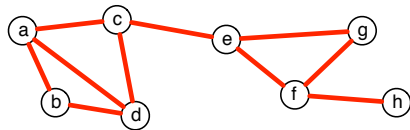
# Unnormalized Laplacian Matrix



$\lambda$	<b>0.278</b>	1.11	
$V$	-0.36	0.08	<i>a</i>
	-0.42	0.18	<i>b</i>
	-0.20	-0.11	<i>c</i>
	-0.36	0.08	<i>d</i>
	0.17	-0.37	<i>e</i>
	0.36	-0.08	<i>f</i>
	0.31	-0.51	<i>g</i>
	0.50	0.73	<i>h</i>
	$v_2$	$v_3$	

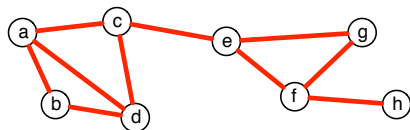


# Laplacian Matrix



$$D^{-1/2} = \begin{pmatrix} 0.577 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.707 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.577 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.577 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.577 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.577 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

# Laplacian Matrix



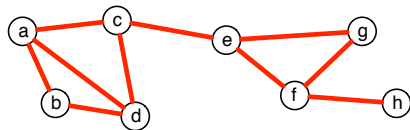
normalized Laplacian

$$L = I - D^{-1/2}AD^{-1/2} =$$

$$\begin{pmatrix} 1 & -0.408 & -0.333 & -0.333 & 0 & 0 & 0 & 0 \\ -0.408 & 1 & 0 & -0.408 & 0 & 0 & 0 & 0 \\ -0.333 & 0 & 1 & -0.333 & -0.333 & 0 & 0 & 0 \\ -0.333 & -0.408 & -0.333 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.333 & 0 & 1 & -0.333 & -0.408 & 0 \\ 0 & 0 & 0 & 0 & -0.333 & 1 & -0.408 & -0.577 \\ 0 & 0 & 0 & 0 & -0.408 & -0.408 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.577 & 0 & 1 \end{pmatrix}.$$

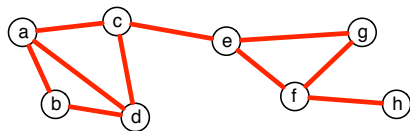


# Laplacian Matrix



eigenvectors of  $L$

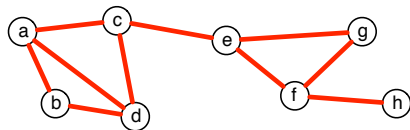
# Laplacian Matrix



eigenvectors of  $L$

$\lambda$	0	<b>0.125</b>	0.724	1.00	1.33	1.42	1.66	1.73
$V$	-.39	0.38	-.09	0.00	0.71	0.26	-.32	0.16
	-.32	0.36	-.27	0.50	0.00	-.51	0.38	-.18
	-.39	0.18	0.36	-.61	0.00	0.03	0.47	-.29
	-.39	0.38	-.09	0.00	-.71	0.26	-.32	0.16
	-.39	-.28	0.48	0.00	0.00	-.57	0.31	0.33
	-.39	-.48	-.29	0.00	0.00	0.05	-.31	-.65
	-.31	-.36	0.27	0.50	0.00	0.51	0.38	-.18
	-.22	-.32	-.61	-.35	0.00	-.07	0.27	0.51

# Laplacian Matrix



$\lambda$	<b>0.125</b>	<b>0.724</b>	
$V$	0.38	-.09	<i>a</i>
	0.36	-.27	<i>b</i>
	0.18	0.36	<i>c</i>
	0.38	-.09	<i>d</i>
	-.28	0.48	<i>e</i>
	-.48	-.29	<i>f</i>
	-.36	0.27	<i>g</i>
	-.32	-.61	<i>h</i>
	$v_2$	$v_3$	

