Hypothesis Testing

$t$-Test and $p$-values

April 13, 2023
Hypothesis Testing

Step 1: Formulate Hypothesis

- Null hypothesis $H_0$: boring
- Specific distribution $f(\theta)$
- Alternative hypothesis $H_1$: interesting

- Null hypothesis $H_0$: $\theta = \theta_0$
- Alternative hypothesis $H_1$: $\theta > \theta_0$

Step 2: Design Experiment

- Random sample $X_1, X_2, \ldots, X_n$
- Choose test statistic $T = T(X_1, \ldots, X_n)$
- Determine threshold critical value at $\alpha$ level

- $P(T \leq \lambda) = 1 - \alpha$
Step 3: Run Experiment

realize sample $x_1, x_2, \ldots, x_n$ (lower case)

calculate $t = \bar{x} \frac{s}{\sqrt{n}}$

↑ actual calculation constant.

Compare $t$ to $t_a$

if $t > t_a$ $\Rightarrow$ reject the null hypothesis

the probability, based on data $x_1, x_2, \ldots, x_n$, that $H_0$ is correct is $\leq \alpha$.

if $t \leq t_a$ $\Rightarrow$ do not reject null hypothesis.
Consider $t_{0.1} \leq t \leq t_{0.05}$

What fraction of experiments would this happen under $H_0$?

\[
Pr\left[ t_{0.1} \leq T \leq t_{0.05} \right]
\]

\[
\neq (0.1 - 0.05) = 0.05
\]
**P-value**: Probability, under $H_0$, that the realized test statistic $t$, or something more extreme (e.g., larger) could occur.

$P_{H_0}(T \leq t) = 1 - p$

$\iff P_{H_0}(T > t) = p$

If $p < 0.05 \iff t > t_{0.05}$
Example

People of Utah are tall compared to dwarves.

$H_0$: people in Utah same as USA $N(\mu, \sigma^2)$

$\sigma^2$ not known

$H_1$: $\mu_{\text{Utah}} > 40$

Random Sample Utah $X_1, \ldots, X_n$ $n = 64$

$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

$s_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

$T = \frac{\bar{X}_n - \mu}{s_n / \sqrt{n}} = \frac{\bar{X}_n - 40}{s_n / \sqrt{8}} \sim t(df = 63)$

Critical value at $\alpha$

$\alpha = 0.05 \Rightarrow t_{\alpha} = qt(1-\alpha, df=63)$
Draw real sample from Utch

\[ \bar{x}_n = 68 \text{ inches} \quad S_n^2 = 36 \text{ in}^2 \]

\[ \frac{\bar{x}_n - 40}{S_n / \sqrt{n}} = \frac{28}{6 / \sqrt{n}} = \frac{4}{3} \cdot 28 = 36.33 = t \]

\[ t_{0.05} \approx 1.97 \]

\[ t > t_{0.05} \implies \text{reject null hypothesis} \]

\[ P\text{-value} = \Pr_{H_0}(T > t) \]

\[ B: 1 - pt(t, df=63) \quad p\text{-value} \leq 1 \cdot 10^{-100} \]
1. \( H_0: \mu = \mu_0 \)
2. Set experiment
   \[ T = \frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} = t(n-1) \]
3. Collect data
   \[ t = T(x_1, \ldots, x_n) \]
   if \( t \geq t_\alpha \) reject \( H_0 \)
   if \( t \leq -t_\alpha \) fail to reject \( H_0 \)

Critical value \( c < \alpha \)

\[ t_\alpha = q_t(1-\alpha, df=n-1) \]