Conditional Probability
Sets

Sample Space \( \mathcal{S} \)

Subsets \( A \leq \mathcal{S} \)

\[ P_r(A) = \frac{|A|}{|\mathcal{S}|} \quad \text{if all } x \in \mathcal{S} \text{ equally likely} \]

Intersection \( A \cap B \) that \( A \) and \( B \) are true

Union \( A \cup B \) \( A \) or \( B \) are true (or both)

Complement \( A^c \) \( A \) is not true.
**Probability Rule**

Inclusion - Exclusion:

\[
\text{Pr}(A \cup B) = \text{Pr}(A) + \text{Pr}(B) - \text{Pr}(A \cap B)
\]

\[
0.2 + 0.7 - 0.1 = 0.8
\]

Not legal:

\[
\text{Pr}(A) = 0.2
\]

\[
\text{Pr}(A \cap B) = 0.3
\]

Rule:

\[
\text{Pr}(C \cup D) = \text{Pr}(C) + \text{Pr}(D) \text{ if } C \cap D = \emptyset
\]
Complement Rule

\[ P(A^c) = 1 - P(A) \]

Difference Rule

\[ P(A - B) = P(A) - P(A \cap B) \]
Conditional Probabilities

Events $A$, $B$

$Pr(A \mid B)$

Probability of $A$, if we know that $B$ is true

$Pr(A \mid B) = \frac{Pr(A \cap B)}{Pr(B)}$
\[
\begin{align*}
\Pr(B) &= 0.5 \\
\Pr(A) &= 0.3 \\
\Pr(A \cap B) &= 0.1 \\
\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
&= \frac{0.1}{0.5} \\
&= \frac{1}{5} = 0.2
\end{align*}
\]
Brain Teaser: 2 coins, fair

\[ B = \text{at least one coin heads} \]
\[ S_B = SHT, TH, HHT \]

\[ A = \text{second coin heads} \]
\[ S_A = HH, HT, TH, TT \]

\[ P(A | B) = \frac{1 \{ HHT \}}{3 \{ SHT, TH, HHT \}} = \frac{1}{3} \]
**Multiplication Rule**

Events A, B

\[ P(A \cap B) = P(A | B) \cdot P(B) \]

\[ P(B), P(A | B) = \frac{P(A \cap B)}{P(B)} \cdot P(B) \]
Tree Diagram: two-stage problems

2 Boxes contain 5 balls \( \{1, 2, 3, 4, 5\} \)

Step 1: choose 1 box (w/ replacement)

Step 2: choose 1 ball from box.

? Prob I choose \( G \)?
Step 1

1/2

1/2

Red

Green

$\Pr(R) = \frac{1}{2}$

$\Pr(G) = \frac{1}{2}$

Step 2

1/3

1/3

1/3

1/3

1/3

$P_r(0) = \frac{1}{16}$

$P_r(1) = \frac{1}{4}$

$P_r(2) = \frac{3}{16}$

$P_r(3) = \frac{3}{16}$

$P_r(0|G) = \frac{1}{14}$

$P_r(0|G) = \frac{1}{14}$

$P_r(1|G) = \frac{3}{14}$

$P_r(2|G) = \frac{3}{14}$

$P_r(3|G) = \frac{3}{14}$
Sampling without Replacement

1 box 10 green balls
10 red balls
draw 2 balls (w/o replace)

Probabilities of 2 red balls?

\( R_1 \) = event that ball 1 red
\( R_2 \) = event that ball 2 red

\[
\Pr(R_2 | R_1) = \frac{9}{19}, \quad \Pr(R_1) = \frac{10}{20} = \frac{1}{2}
\]

\[
\Pr(R_1 \land R_2) = \Pr(R_2 | R_1) \cdot \Pr(R_1) = \frac{9}{19} \cdot \frac{1}{2} = \frac{9}{38}
\]
Pool of 3 rad balls, using replacement:

10 red, 10 green

\[ P(R_1 \cap R_2 \cap R_3) = \frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} = \frac{1}{120} \]

\[ P(R_1) - P(R_2), P(R_3) \]

\[ P(R_1 \cap R_2) = \frac{3}{10} \times \frac{2}{9} = \frac{1}{15} \]

\[ P(R_3) - P(R_1 \cap R_2) = \frac{3}{8} - \frac{1}{15} = \frac{39}{120} \]

\[ P(R_1 | R_2) = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{\frac{3}{10}}{\frac{2}{9}} = \frac{27}{20} \]

\[ P(R_3 | R_2) = \frac{P(R_2 \cap R_3)}{P(R_2)} = \frac{\frac{3}{8}}{\frac{2}{9}} = \frac{27}{16} \]

\[ P(R_1 | R_2 \cap R_3) = \frac{P(R_1 \cap R_2 \cap R_3)}{P(R_2 \cap R_3)} = \frac{\frac{3}{10}}{\frac{2}{9}} = \frac{27}{20} \]

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