FoDA L9

Linear Algebra Review #2

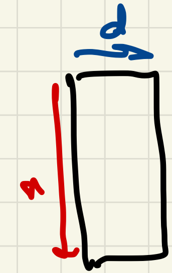
Norms, Linear Independence, Rank

Sep 20, 2022

Linear Algebra

vectors & Matrices

$u, v, a_i \in \mathbb{R}^d$ $\quad A, B \in \mathbb{R}^{n \times d}$

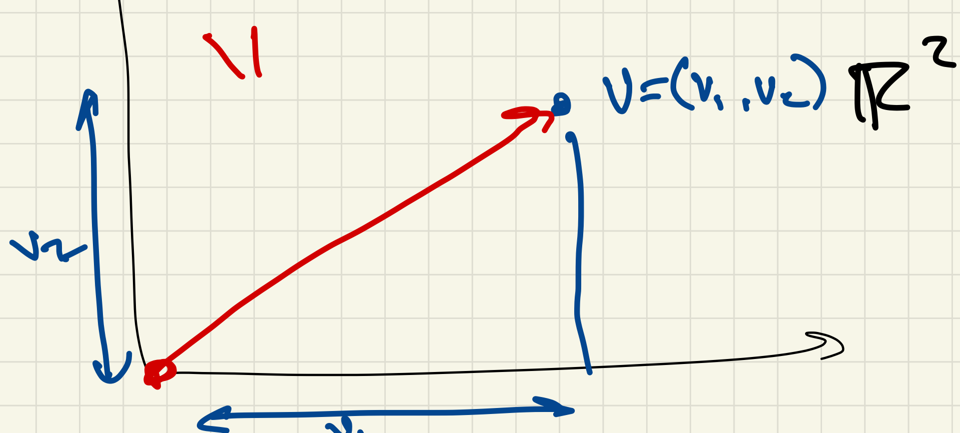


$M \in \mathbb{R}^{d \times d}$

$u = (u_1, u_2, \dots u_d)$

Norm

$\|v\| = \sqrt{v_1^2 + v_2^2}$



p- Norm

$v \in \mathbb{R}^d$ $\quad$ $p \in [1, \infty)$

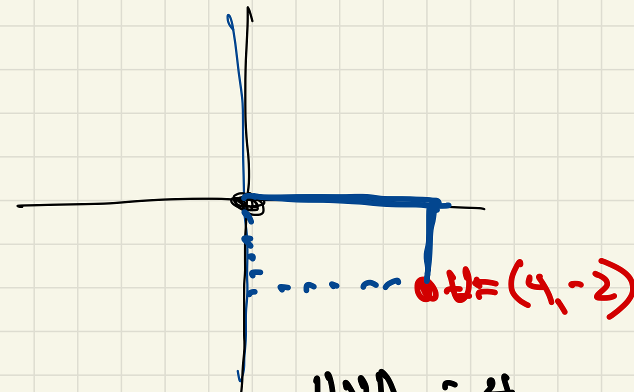
$$\|v\|_p = \Big(\underbrace{\sum_{j=1}^{d} |v_j|^p}_{(\text{unit})^p}\Big)^{1/p}$$

keep units

$$\left((\text{unit})^p\right)^{1/p} = \text{unit}$$

standard $\|v\|_2 = \|v\|$

$$\|v\|_1 = \left(\sum_{j=1}^{d} |v_j|^1\right)^{1/1} = \sum_{j=1}^{d} |v_j|$$



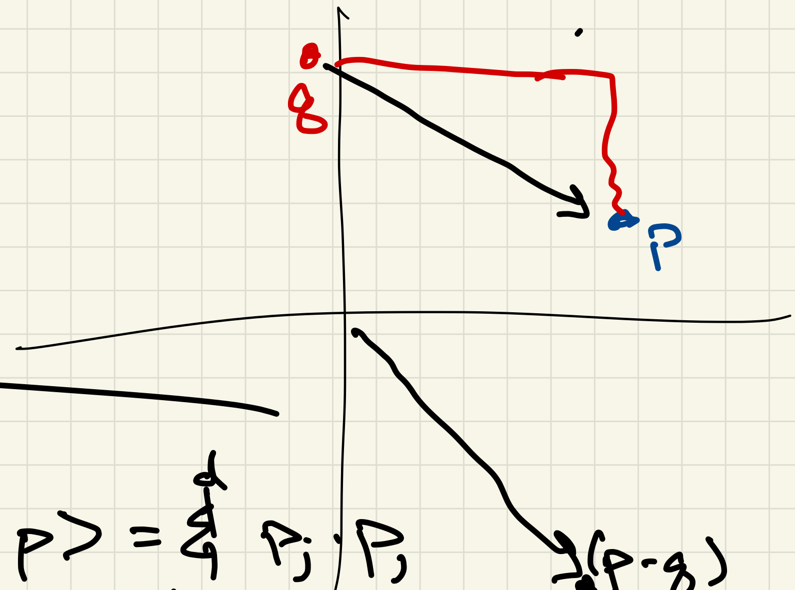
$\|v\|_\infty = 4$

$$\|v\|_\infty = \max_{j=1..d} |v_j|$$

Distance between high-dimensional points

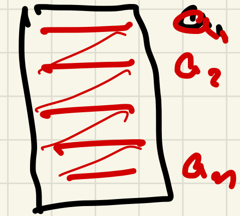
$$p, q \in \mathbb{R}^d$$

$$d(p,q) = \|p-q\|$$



$$\left(\|p\|_2\right)^2 = \sum_{j=1}^{d} (p_j)^2 = \langle p, p \rangle = \sum_{j=1}^{d} p_j \cdot p_j$$

Matrix Norms



$A \in \mathbb{R}^{n \times d}$

Frobenius $\quad \|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{d} (A_{ij})^2}$

$$= \sqrt{\sum_{i=1}^{n} \|a_i\|_2^2}$$

Spectral $\quad \|A\|_2 = \max_{\substack{x \in \mathbb{R}^d \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\substack{y \in \mathbb{R}^n \\ y \neq 0}} \frac{\|y^T A\|_2}{\|y\|_2}$

$$\|A\|_F \geq \|A\|_2$$

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$\|A\|_F^2 = 9 + 49 + 4 + 1 + 4 + 25$$

$$= (9.59)^2$$

$$= 92$$

$$\|A\|_F = 9.59 = \sqrt{\|A\|_F^2}$$

Linear Independence

k vectors $\quad x_1, x_2, \dots x_k \in \mathbb{R}^d$

k scalar $\quad \alpha_1, \alpha_2, \dots \alpha_k \in \mathbb{R}$

$$z = \underbrace{\sum_{i=1}^{k} \alpha_i x_i}_{} \in \mathbb{R}^d$$

Set $X = \{x_1 \dots x_k\}$

$$\text{span}(X) = \left\{ z \;\middle|\; z = \sum_{i=1}^{k} \alpha_i x_i,\ \alpha_i \in \mathbb{R} \right\}$$

such that

if $\text{span}(X) = \mathbb{R}^d \Rightarrow X$ a basis

$$X = \{x_1, x_2\} \qquad x_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \in \mathbb{R}^3$$

linearly dependent on X

$$z_1 = 1x_1 - 2x_2 = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$$

$\alpha_1 = 1 \qquad \alpha_2 = -2$

linearly independent of X

$$z_2 = \begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix} = \underset{1}{\alpha_1} x_1 + \underset{1}{\alpha_2} x_2 = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \neq z_2$$

$\{x_1, x_2, z_1\}$ linearly dependent

$\{x_1, x_2, z_2\}$ linearly independent

Set vectors $X = \{x_1 \dots x_k\} \subset \mathbb{R}^d$

$z \in \mathbb{R}$ is linearly indepent of X

if $\nexists$ $\alpha_1 \dots \alpha_k$ so $z = \sum_{j=1}^{k} \alpha_j x_j$

there does not exist

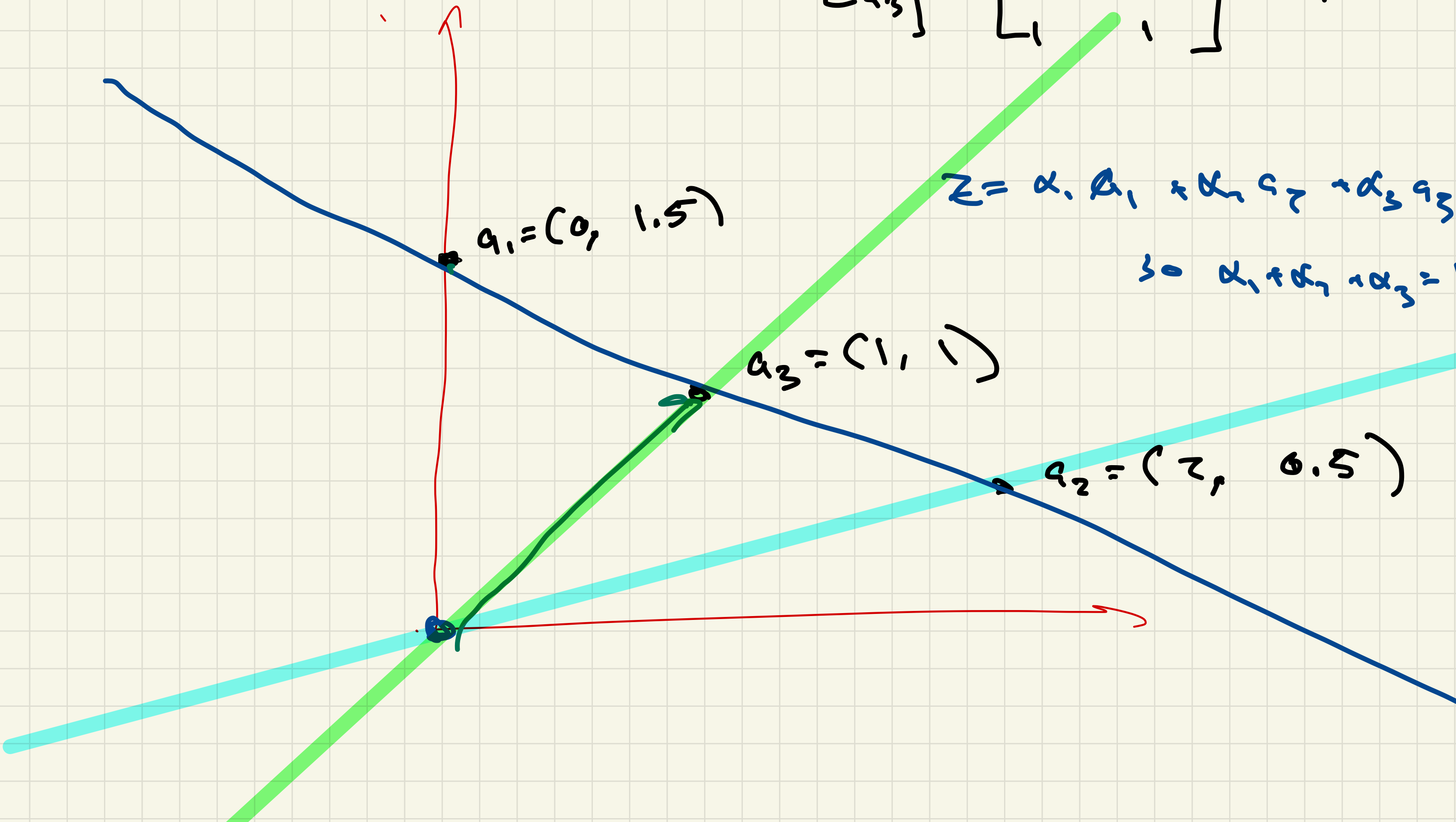
otherwise z linearly dependent

X linearly idependent if for each $j \in X$

no $\alpha_1 \dots \alpha_{j-1}, \alpha_{j+1} \dots \alpha_k$ so

$$x_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i x_i$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1.5 \\ 2 & 0.5 \\ 1 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$



$$z = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

$$\text{s.t. } \alpha_1 + \alpha_2 + \alpha_3 = 1$$

Rank of Matrix (of set of vectors)

$X \in \mathbb{R}^{n \times d}$ $\quad X = \{x_1, \dots x_n\} \subset \mathbb{R}^d$

→ maximum number of linearly independent vectors in X.

$\text{rank}\{x_1, x_2, z_1\} = 2$ $\qquad$ $\text{rank}\{x_1, x_2, z_2\} = 3$

$\text{rank}(A) = \text{rank}(A^T)$

A is full rank if $\text{rank}(A) = \min\{n, d\}$

$\text{rank}(A) \leq \min\{n, d\}$