FoDA : Linear Algebra #3

L10 : Square Matrices

Rank of a matrix $A \in \mathbb{R}^{n \times d}$

rows $a_1, a_2 \dots a_n$ of A. $\quad a_i \in \mathbb{R}^d$

vector

$z \in \mathbb{R}^d \qquad z = \sum_{i=1}^{n} \alpha_i a_i$

scalar

linearly dependent on A $\longrightarrow$ $\qquad z \in \text{Span}(a_1, \dots a_n)$

$\text{span}(A)$

rank(A) is maximum number of rows which are linearly independent of each other.

$\text{rank}(A) \leq \min\{n, d\}$

Example

$$A = \begin{bmatrix} 3 & -7 & 2 \\ -1 & 2 & -5 \end{bmatrix}$$

$$\text{rank}(A) \leq \min\{2,3\}$$
$$\leq 2$$

$$(3, -7, 2) = \overset{?}{\alpha}\,(-1, 2, -5)$$

$$-\frac{1}{3}$$

$$\text{rank}(A) = 2$$

$$G = \begin{bmatrix} 1 & 3 \\ 4 & 7 \end{bmatrix} \qquad \text{rank}(G) = 2$$

full rank $= \text{rank}(A) = \min\{n, d\}$

Square Matrix

$M \in \mathbb{R}^{n \times n} \equiv$ same # rows, and columns.

Inverse of a Matrix: M^{-1}

multiply by $z^{-1} \equiv$ divide by z

only do if M is square and full rank

$$(M)(M^{-1}) = I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$(M^{-1})(M) = I$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Eigenvectors & Eigenvalues

$M \in \mathbb{R}^{n \times n}$

$$M v = \lambda v$$

$v \in \mathbb{R}^n$

$\lambda \in \mathbb{R}$

v eigenvector

λ eigenvalue

make $\|v\|_2 = 1$

unit vectors

at most n distinct eigenvector (and eigenvalue)

$$M = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 5 & 2 \\ 4 & 2 & 8 \end{bmatrix} \in \mathbb{R}^{3\times3}$$

$$v_1 = \begin{bmatrix} 0.43 \\ 0.44 \\ 0.78 \end{bmatrix} \qquad \lambda_1 = 11.36 = \|M\|_2^2$$

first, top eigenvector
one w/ largest eigenvalue

$$v_2 = \begin{bmatrix} -0.11 \\ -0.83 \\ 0.54 \end{bmatrix} \qquad \lambda_2 = 4.10$$

$$v_3 = \begin{bmatrix} -0.90 \\ 0.31 \\ 0.31 \end{bmatrix} \qquad \lambda_3 = -0.46$$

eigenvalues
(in class)

λ_i – real,
(positive or zero)

Square Matrix M w/ n positive, real eigenvalues

↳ positive definite

Also have $x \in \mathbb{R}^n$

$x^T M x \geq 0$

for all x

if eigenvalues are real, non-negative $(x^T A)(A^T x)$

↳ positive semidefinite

matrix $A \in \mathbb{R}^{n \times d}$

$M = AA^T \in \mathbb{R}^{n \times n}$

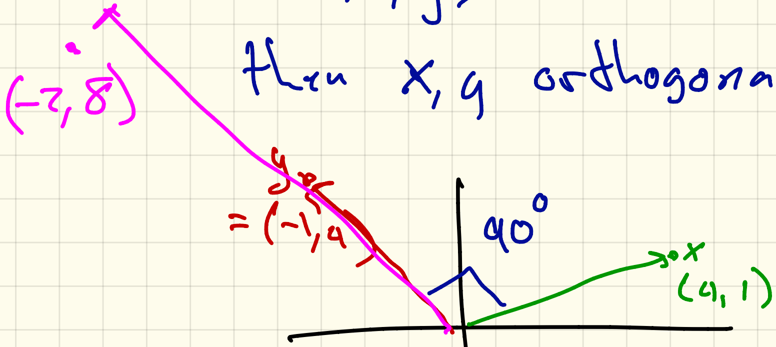
↳ must be positive semidefinite

M also full rank → positive definite

Orthogonality

$x, y \in \mathbb{R}^d$ if $\langle x, y \rangle = 0$

then x, y orthogonal



$x = (2, -3, 4, -1, 6)$

$y = (4, 5, 3, -7, -2)$

$8 - 15 + 12 + 7 - 12 = 0$

↳ orthogonal

$\langle (4,1), (-1,4) \rangle$

$-4 + 4 = 0$

$\langle (4,1), (-2,8) \rangle$

$-8 + 8 = 0$

Matrix $V \in \mathbb{R}^{n \times d}$

has all orthogonal columns

unit vectors

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_d \\ | & | & & | \end{bmatrix}$$

$\hookrightarrow$ orthonormal

$$\|v_i\| = 1$$

$$\langle v_i, v_j \rangle = 0$$

Square Matrix $U \in \mathbb{R}^{n \times n}$

and all orthogonal rows & columns

then orthogonal matrix

also rows / columns unit vectors

Orthogonal Matrix $U \in \mathbb{R}^{n \times n}$

Then $UU^T = I$

$$\langle u_i, u_i \rangle = 1 = \|u_i\|^2$$

$$U^T = U^{-1} \qquad \langle u_i, u_j \rangle = 0$$

$$I = \begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & 0 \\ 0 & & & & 1 \end{bmatrix}$$

Eigen vectors

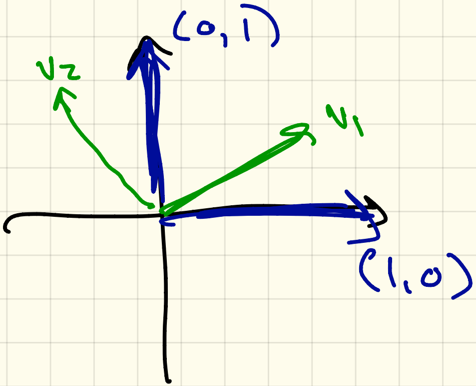
must be orthogonal

$$M v_i = v_i \lambda$$

$$M v_j = v_j \lambda$$

then $\langle v_i, v_j \rangle = 0$

$v_i \neq v_j$



Orthogonal Matrices

have no scale information

any vector $x \in \mathbb{R}^n$

orthogonal matrix $M \in \mathbb{R}^{n \times n}$

$$\|x\| = \|Mx\|$$