L24: Markov Chains

Jeff M. Phillips

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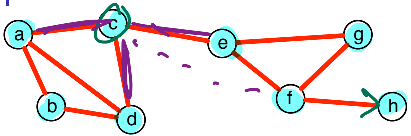
Markov Chain : Life Lessons

- [**L1**] *Only your current position matters going forward, don't worry about the past.*
- [**L2**] *You just need to worry about one step at a time; you will get there eventually (or you won't).*
- [**L3**] *In the limit, everyone has perfect karma.*

Graphs



Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$$

Matrix-Style: As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

adjacency

$$G = \begin{array}{c|cccccccc} & a & b & c & d & e & f & g & h \\ \hline a & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ b & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ c & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ f & 0 & 0 & \cancel{0} & 0 & 1 & 0 & 1 & \boxed{1} \\ g & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} = A$$

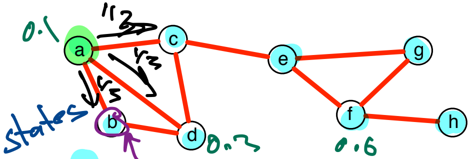
Markov Chain

Δ_n = vectors in $\mathbb{R}^n$

$v \in \Delta_n$

$v_j \in [0,1]$

$\sum_j v_j = 1$



(V, P, q): V node set, P probability transition matrix, q initial state.

e.g. $q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$ or $q^T = [0.1\ 0\ 0\ 0.3\ 0\ 0.6\ 0\ 0]$. $\in \Delta_n$

$$P = \begin{array}{c} \text{a} \\ \left(\begin{array}{cccccccc}
0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\
1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0
\end{array}\right) \end{array}$$

probability $\frac{A_i}{\|A_i\|_1} \in \Delta_n$ $\quad \|v\|_1 = \left(\sum_{j=1}^{n} |v_j|^1\right)^{1/1}$

Transitions

b start

$$P = \begin{array}{c} a \\ \\ \\ d \\ \\ \\ \\ \\ \end{array}\begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Transitions

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} a & & & d & & & & \end{array} \\
P = \begin{array}{c} a \\ \\ \\ \\ \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\
1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\
0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\
0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0
\end{pmatrix}
\end{array}
\text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]
$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T .$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T .$$

$\frac{1}{2} \cdot \frac{1}{3}$

$\frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3}$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

In the limit: $q_n = P^n q$

Not

uniform $= \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

In the limit: $q_n = P^n q$

Markov

[**L1**] *Only your current position matters going forward, don't worry about the past.*

Two versions of Markov Chain

1. Only consider 1 state at a time

$$g_j = [0, 0, 1, 0, 0, 0, \dots]$$

2. Probability Distribution as state

ex. $g_j = [\frac{1}{3}, 0, \frac{1}{2}, 0, 0, \frac{1}{12}, \frac{1}{12}, 0] \in \Delta_n$

Limiting state q_n as $n \to \infty$

$$q_n = P^n q_0$$

limit does not always exist!

Ergodic (Markov Chain) $\rightarrow$ then limit exists

MC is ergodic if $\exists\, t$ so that

\# steps

for all $n \geq t$ $\quad q_n(j) > 0$ strictly greater than 0.

for all $j \in V$

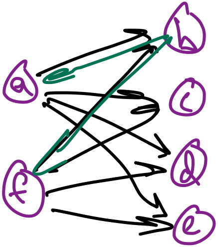
Not ergodic iff

① has absorbing + transient states

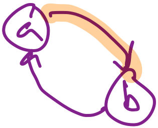
② disconnected.

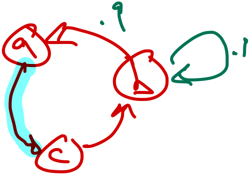
③ cyclic

Cyclic Examples



bipartite



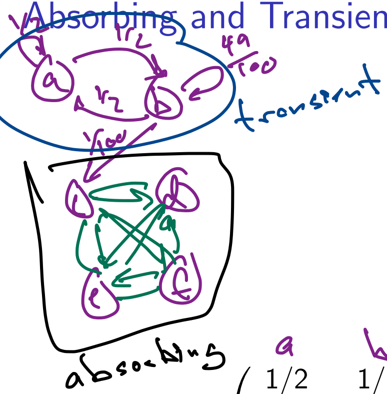


$$\begin{array}{cc} a & b \\ \hline \end{array}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{pmatrix} 0 & 1\;0.9 & 0 \\ 0 & 0\;.1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

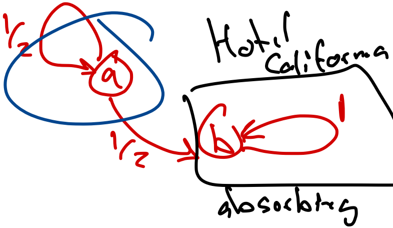
$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \end{pmatrix}$$

Absorbing and Transient Examples

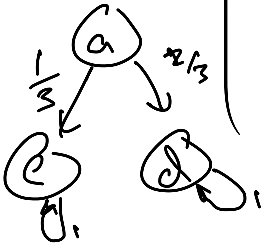


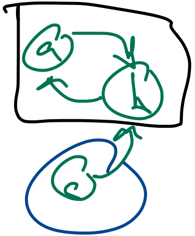
$$\begin{array}{cc} a & b \\ \left(\begin{array}{c} 1/2 \\ 1/2 \end{array}\right. & \left.\begin{array}{c} 0 \\ 1 \end{array}\right) \end{array}$$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$



$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 49/100 & 0 & 0 & 0 & 0 \\ 0 & 1/100 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

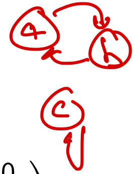




Unconnected Examples

$$\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$



$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Limiting State

if MC is ergodic

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.

any initial condition

same q_*

Limiting State

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.

[L2] *You just need to worry about one step at a time; you will get there eventually (or you won't).*

not ergodic

Delicate Balance

eigenvectors of P

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.
Also $q_* = P\underbrace{\boxed{P^* q}}_{q_*}$ thus $q_* = P q_*$.

So the probability of (being in a state i and leaving to j) is the same as (being in another state j and arriving in i)

$$P_{i,j} q_*(i) = P_{j,i} q_*(j)$$

probability in state i and transition to state j = Probability in state j and transition to state i

Delicate Balance

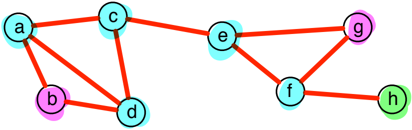
Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.
Also $q_* = PP^* q$ thus $q_* = Pq_*$.

So the probability of (being in a state i and leaving to j) is the same as (being in another state j and arriving in i)

$$P_{i,j} q_*(i) = P_{j,i} q_*(j)$$

[**L3**] *In the limit, everyone has perfect karma.*

Limiting State



In our example

$$
\begin{aligned}
q_* &= (0.15, 0.1, 0.15, 0.15, 0.15, 0.15, 0.1, 0.05) \\
&= \left(\frac{3}{20}, \frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{20}\right)
\end{aligned}
$$

Algorithms for computing q^*

(1) $\mathrm{eig}(P) \Longrightarrow$ top eigenvalue $v_1 \in \mathbb{R}^{n \times 1}$

$$q^* = \frac{v_1}{\|v_1\|_2}$$

(2) $q_n = P(P(P(\ldots P(q_0)\ldots)))$ n steps, n large

Power ~~method~~ method.

(3) $P^n = P \cdot P \cdots P$

$((P^2 \cdot P^2)^2)^2 \ldots \log n$ steps

(4) Random walk of P Keep track of distribution

Metropolis Algorithm

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Metropolis Algorithm

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Metropolis on V and w

Initialize $v_0 = [0\ 0\ 0\ \ldots\ 1\ \ldots\ 0\ 0]^T$.
repeat
  Generate $u \sim K(v, \cdot)$
  if $(w(u) \geq w(v_i))$ **then**
    Set $v_{i+1} = u$
  else
    With probability $w(u)/w(v)$ set $v_{i+1} = u$
  else
    Set $v_{i+1} = v_i$
until "converged"
return $V = \{v_1, v_2, \ldots, \}$