L21: Markov Chains

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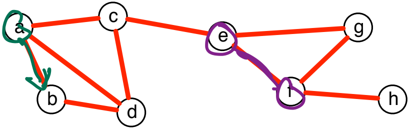
Markov Chain : Life Lessons

- [**L1**] *Only your current position matters going forward, don't worry about the past.*
- [**L2**] *You just need to worry about one step at a time; you will get there eventually (or you won't).*
- [**L3**] *In the limit, everyone has perfect karma.*

Graphs



Mathematically: $G = (V, E)$ where

$V = \{a, b, c, d, e, f, g\}$ and

$$E = \Big\{\{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{c, e\}, \{e, f\}, \{e, g\}, \{f, g\}, \{f, h\}\Big\}.$$

Matrix-Style: As a matrix with 1 if there is an edge, and 0 otherwise. (For a directed graph, it may not be symmetric).

A_i $\qquad$ A

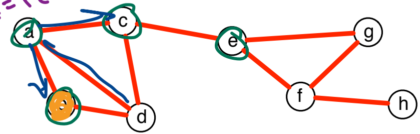
$G =$

	a	b	c	d	e	f	g	h
a	0	1	1	1	0	0	0	0
b	1	0	0	1	0	0	0	0
c	1	0	0	1	1	0	0	0
d	1	1	1	0	0	0	0	0
e	0	0	1	0	0	$\boxed{1}$	1	0
f	0	0	0	0	$\boxed{1}$	0	1	1
g	0	0	0	0	1	1	0	0
h	0	0	0	0	0	1	0	0

$$= \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Markov Chain

$P_1 = P(a)$



q_0

(V, P, q): V node set, P probability transition matrix, q initial state.

e.g. $q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$ or $q^T = [0.1\ 0\ 0\ 0.3\ 0\ 0.6\ 0\ 0]$.

$P_1 = P(a)$

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix}$$

$P_1 = \Delta_n$ probability vector

$P_{ij} = \text{Prob}(\dots j)$

$P_1 = \frac{A_1}{\|A_1\|_1}$

$\|A_1\|_1 = \left(\sum_{j=1}^{n} |A_{1j}|^1\right)^{1/1}$

Transitions

$$P = \begin{array}{c} \begin{array}{cccccccc} & b & & & & & & \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \end{array} \text{ and } q^T = \overset{b}{\boxed{[0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]}}$$

$$q_1 = \boxed{Pq} = \boxed{\begin{bmatrix} \overset{a}{\frac{1}{2}} & 0 & 0 & \overset{d}{\frac{1}{2}} & 0 & 0 & 0 & 0 \end{bmatrix}}^T.$$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix}$$

and $q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$

1/2 1/2

one step

$$\boxed{q_1 = Pq} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = \boxed{Pq_1} = PPq = \boxed{P^2} q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

Transitions

q_2

q_3 $\quad P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix}$ and $q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T. = P^3 q_0$$

Transitions

cols normalised.

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

In the limit: $q_n = P^n q$ $= P(P\underbrace{\ldots(Pq)\ldots}_{n \text{ times}}))$

NOT ~~uniform~~

uniform? $q_n = \left(\frac{1}{|V|}, \frac{1}{|V|}, \ldots \frac{1}{|V|}\right)$

Transitions

$$P = \begin{pmatrix} 0 & 1/2 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 & 1/3 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 & 0 \end{pmatrix} \text{ and } q^T = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$q_1 = Pq = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_2 = Pq_1 = PPq = P^2 q = \begin{bmatrix} \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

$$q_3 = Pq_2 = \begin{bmatrix} \frac{1}{3} & \frac{1}{9} & \frac{1}{9} & \frac{1}{3} & \frac{1}{9} & 0 & 0 & 0 \end{bmatrix}^T.$$

In the limit: $q_n = P^n q$

[**L1**] *Only your current position matters going forward, don't worry about the past.*

Two ways of thinking of Markov Chain

(1) Only considers 1 possible state (node) at a time
(e.g. $g_i = [0, 0, 1, 0, 0, 0]$

(2) Probability Distribution on states (nodes)
(e.g. $g_0 = [\frac{1}{3}, 0, 0, \frac{1}{6}, 0, \frac{1}{2}, 0]$

Limiting state q_n as n goes to ∞ in limit

$q_n = P^n q_0$

Ergodic (Markov Chain)

(when limit exists)

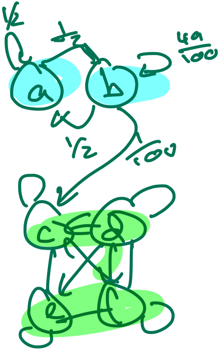
MC is ergodic if $\exists\ t$ (# steps) so that for all $n \geq t$ $q_n(j) > 0$. for all $j \in V$

Not ergodic iff

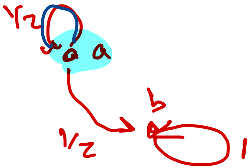
1. has absorbing & transient states
2. disconnected.
3. cyclic.

Absorbing and Transient Examples

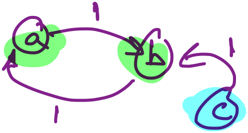
$P_a^n\left(\frac{1}{2}n\right)$



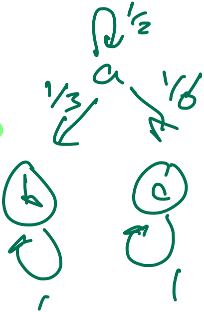
$$\begin{array}{c|cc} & a & b \\ \hline a & 1/2 & 0 \\ b & 1/2 & 1 \end{array}$$



$$\begin{array}{ccc} a & b & c \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}$$



$$\begin{array}{cccccc} a & b & c & d & e & f \\ \hline 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 49/100 & 0 & 0 & 0 & 0 \\ 0 & 1/100 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \end{array}$$



No absorbing + transient if can get from any node to any other.

Unconnected Examples

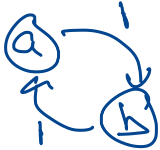
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/2 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Cyclic Examples

most common

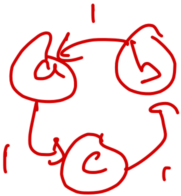


$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

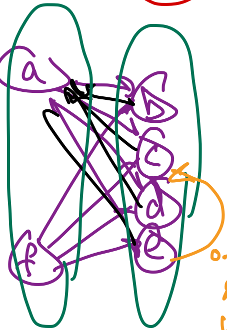
$q_0 = [1, 0]$

$q_1 = P q_0 = [0, 1]$

$q_{17} = [0, 1]$



$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 0 & 1/4 \\ 0 & 1/2 & 1/2 & 1/2 & 1/2 & 0 \end{pmatrix}$$

0.001

breaks cyclic structure.

Limiting State

if MC ergodic

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.

initial state q_0 does not matter.

Limiting State

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.

[L2] *You just need to worry about one step at a time; you will get there eventually (or you won't).*

if has absorbing and transient states

(maybe not)

Delicate Balance

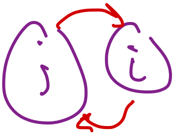
limiting state

$q_* \in \Delta_{|V|}$

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.
Also $q_* = \boxed{P}\underbrace{P^* q}_{q_*}$ thus $\boxed{q_* = P q_*}$.

So the probability of being in a state i and leaving to j is the same as being in another state j and arriving in i

$$\boxed{P_{i,j} q_*(i) = P_{j,i} q_*(j)}$$



probability in state i

probability of moving to state j given that in state i

Delicate Balance

$q_* = P q_*$

q_* eigenvector of P.

Let $P^* = P^n$ as $n \to \infty$.
Let $q_* = P^* q$.
Also $q_* = PP^* q$ thus $\boxed{q_* = Pq_*}$.

So the probability of being in a state i and leaving to j is the same as being in another state j and arriving in i

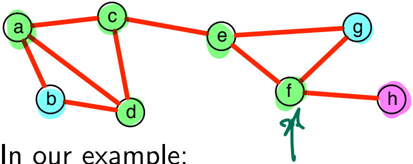
$$P_{i,j} q_*(i) = P_{j,i} q_*(j)$$

[L3] *In the limit, everyone has perfect karma.*

Limiting State

not

uniform



In our example:

$$q_* = (0.15, 0.1, 0.15, 0.15, 0.15, 0.15, 0.1, 0.05)$$
$$= (\frac{3}{20}, \frac{1}{10}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{3}{20}, \frac{1}{10}, \frac{1}{20})$$

f

Algorithms for computing q_*

1. $\mathrm{eig}(P) \Rightarrow$ top eigenvector $v_1 \in S^{m-1}$

$$q_* = \frac{v_1}{\|v_1\|}$$

2. $q_n = P(P \ldots P(q_0)) \ldots)$ n steps, n large

Power method.

3. $q_n = P^n q_0$ for large enough

$n = 2^k$ $\quad P^2 \to P^4 \to P^8 \to \ldots P^{2^k}$

4. Random walk on P

Keep track of where I walk.

Metropolis Algorithm

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Metropolis Algorithm

Metropolis, Rosenbluth, Rosenbluth, Teller, and Teller in 1953

Metropolis on V and w

Initialize $v_0 = [0\ 0\ 0\ \ldots\ 1\ \ldots\ 0\ 0]^T$.
repeat
  Generate $u \sim K(v, \cdot)$
  if $(w(u) \geq w(v_i))$ **then**
    Set $v_{i+1} = u$
  else
    With probability $w(u)/w(v)$ set $v_{i+1} = u$
  else
    Set $v_{i+1} = v_i$
until "converged"
return $V = \{v_1, v_2, \ldots,\}$