From Parametricity to Conservation Laws, via Noether's Theorem

Written by Robert Atkey

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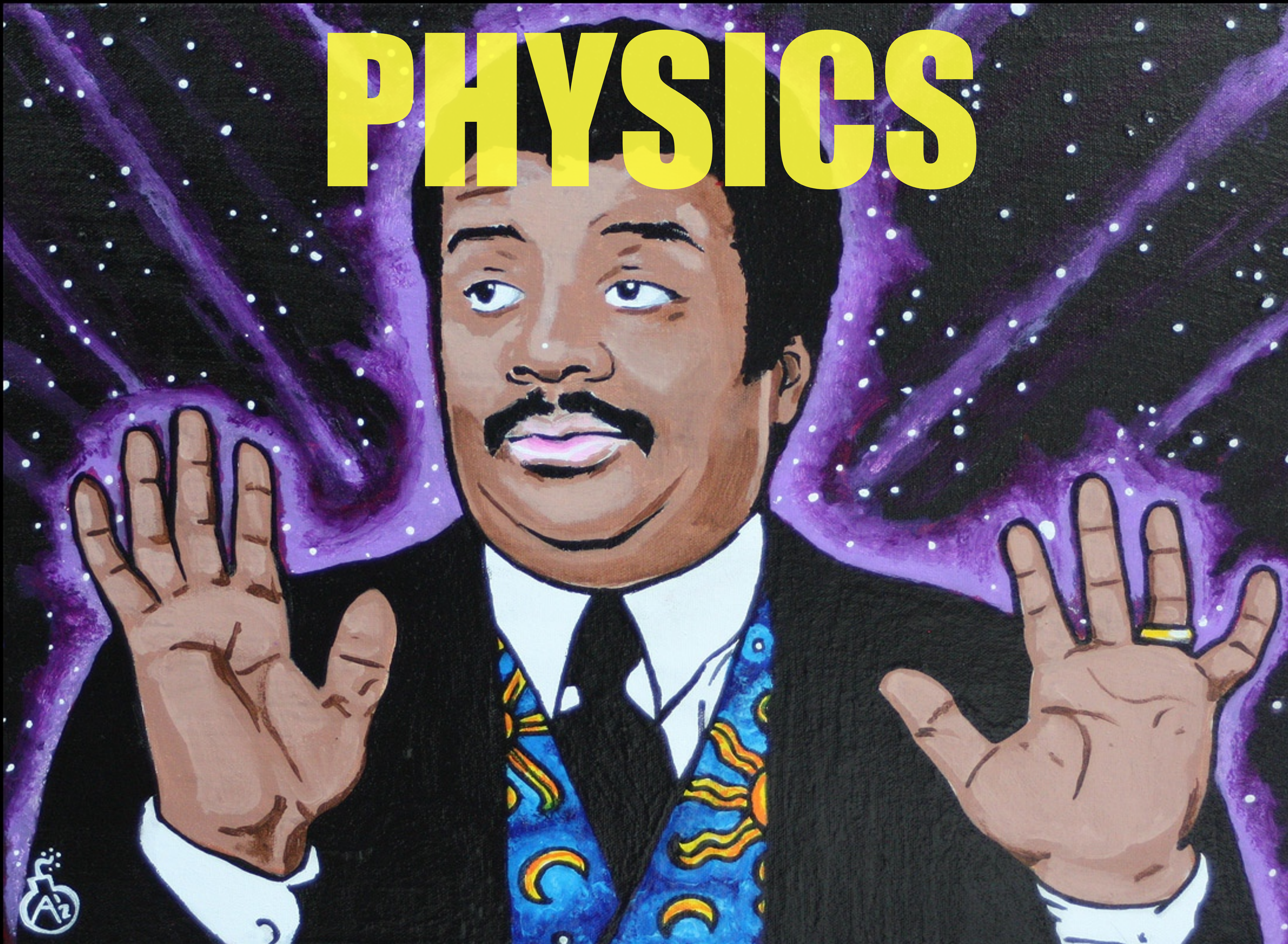
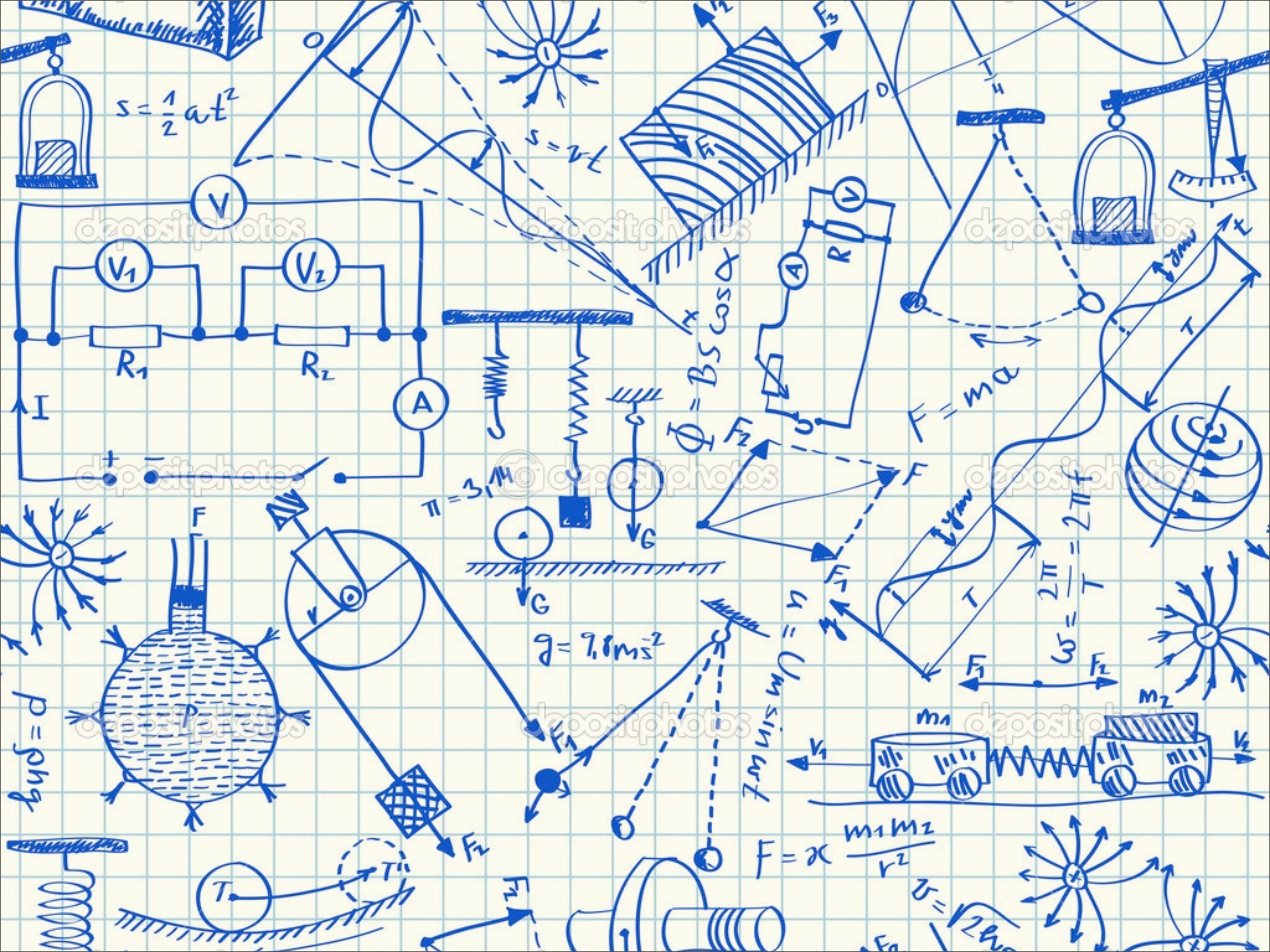
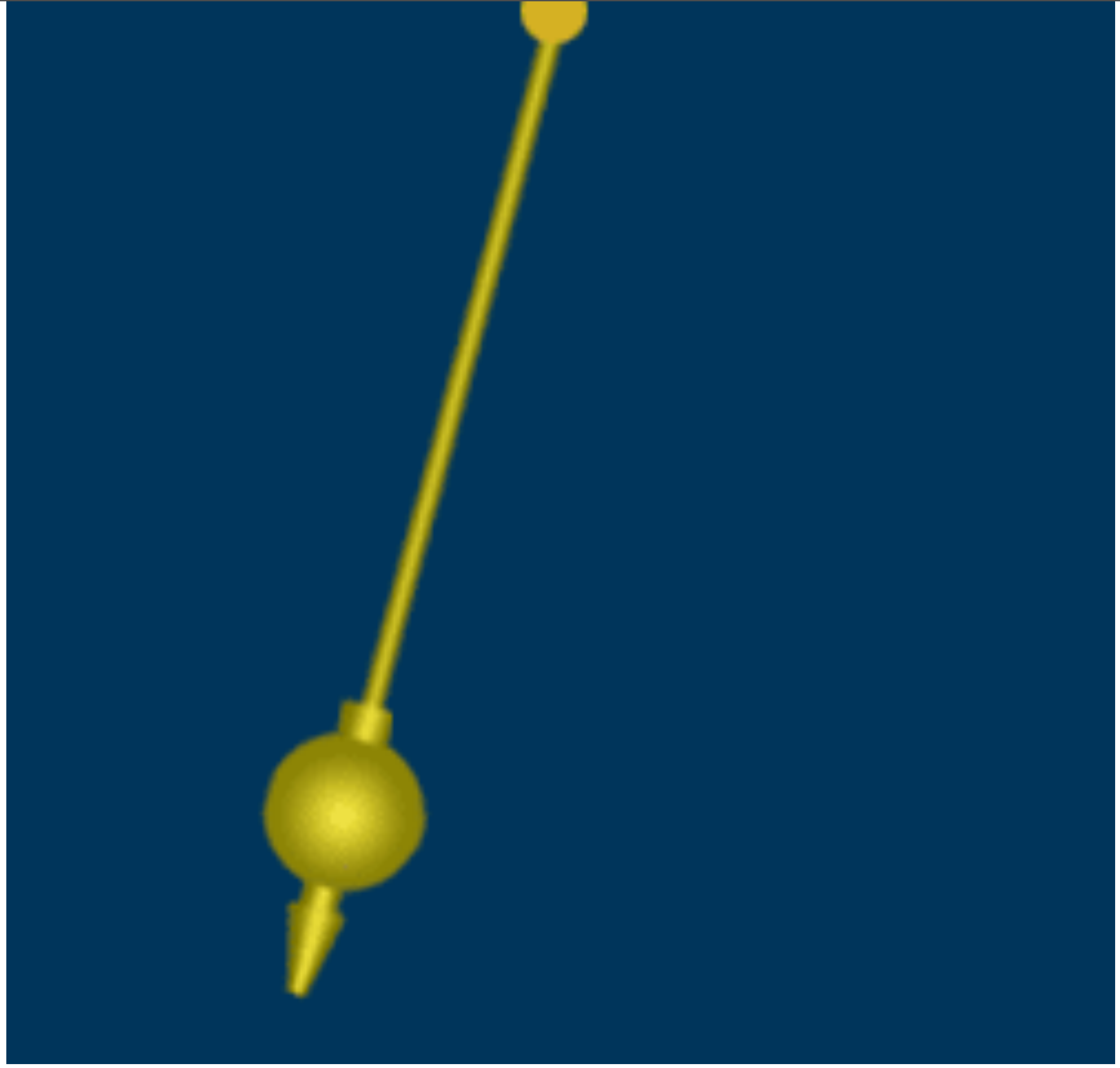
Presented by Ben Carriel & Ben Greenman

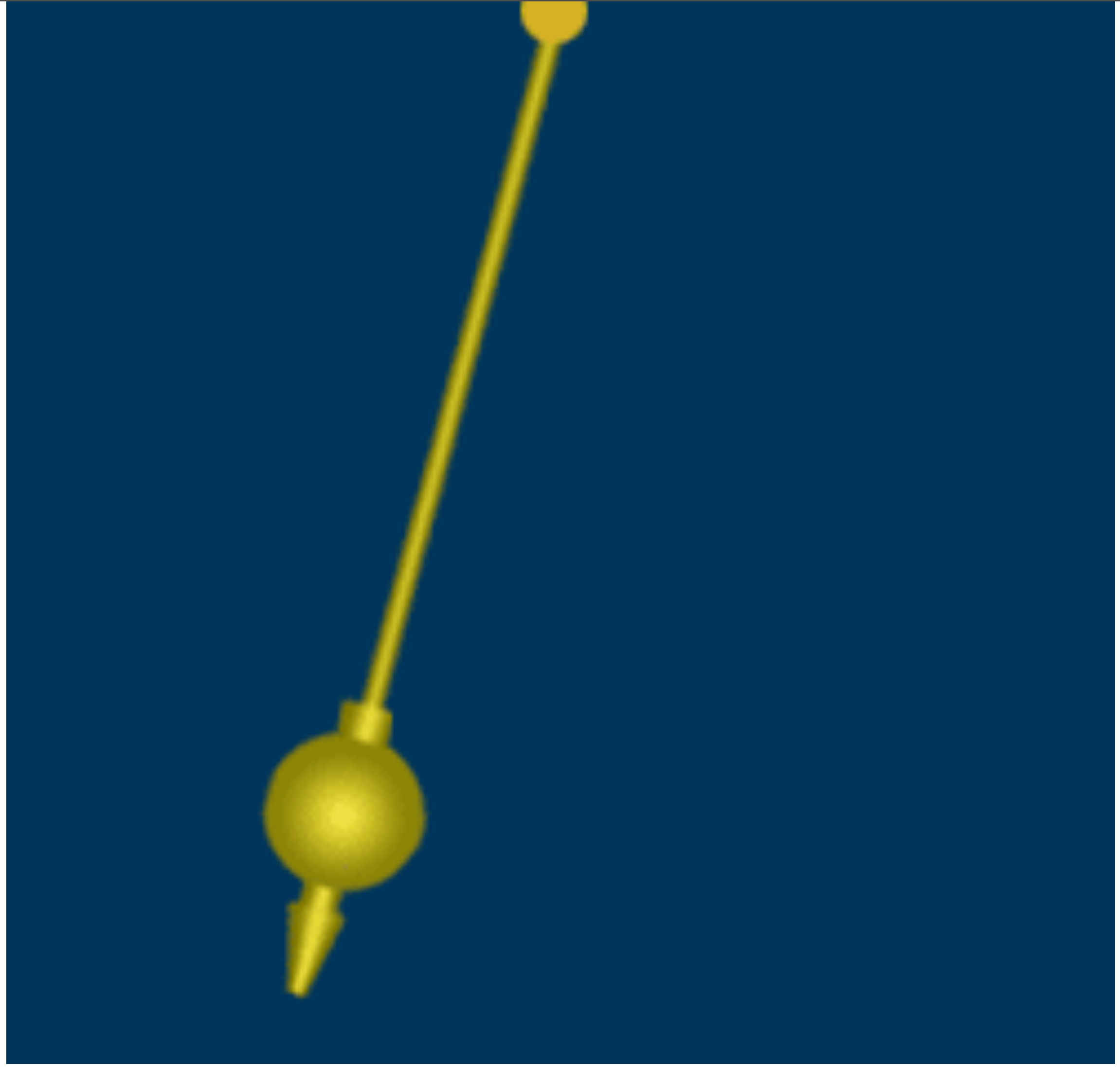
2014-04-07


PHYSICS

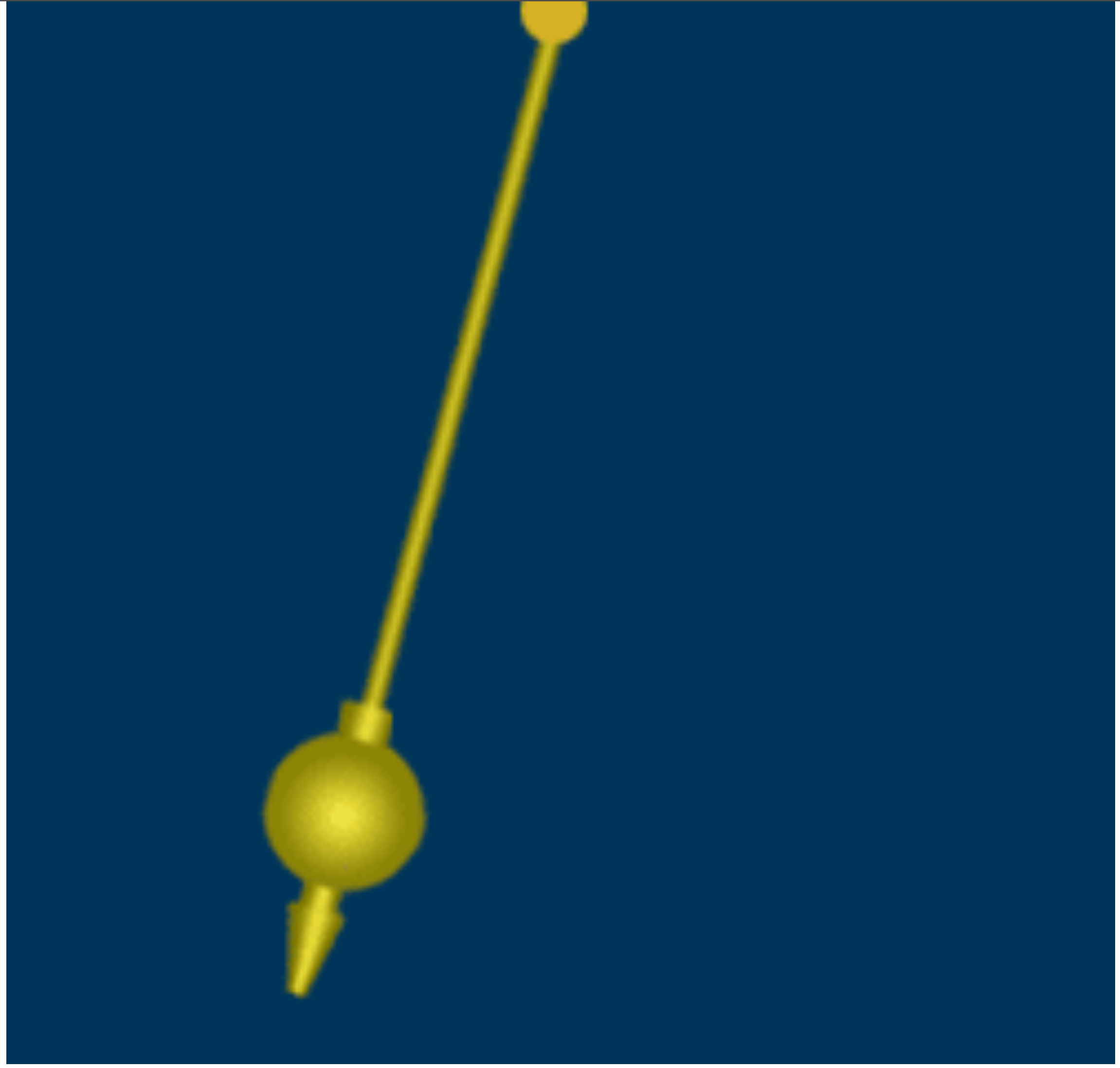


$s = \frac{1}{2}at^2$
$s = vt$
V
V_1
V_2
R_1
R_2
A
I
+
−
F
P
$p = \rho gh$
$\pi = 3,14$
G
$g = 9,8 ms^{-2}$
F_1
F_2
F_3
$\Phi = BS\cos\alpha$
R
$F = ma$
$\omega = \frac{2\pi}{T} = 2\pi\nu$
T
y_m
m_1
m_2
v_1
v_2
$F = \varkappa \frac{m_1 m_2}{r^2}$
depositphotos



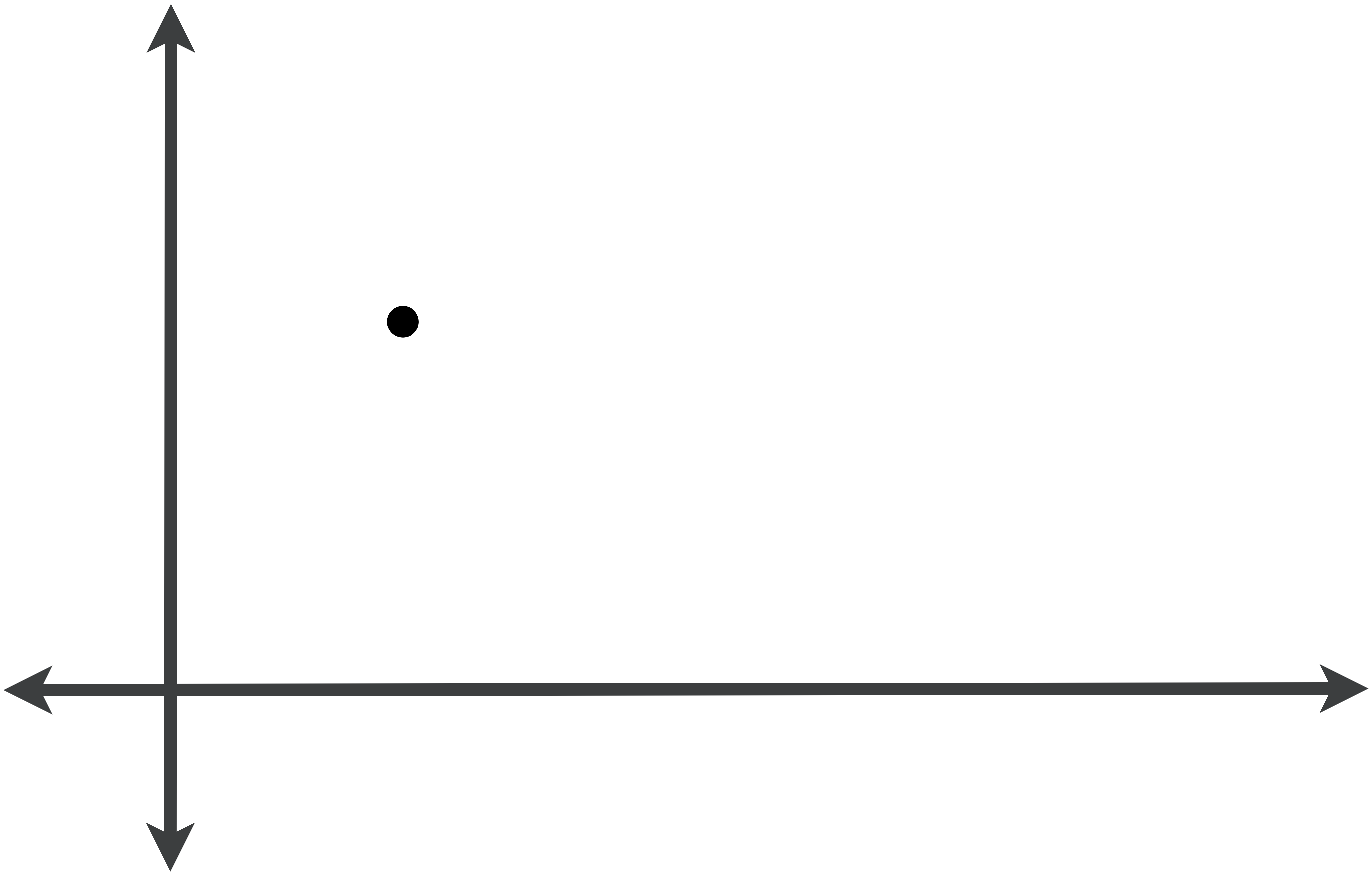


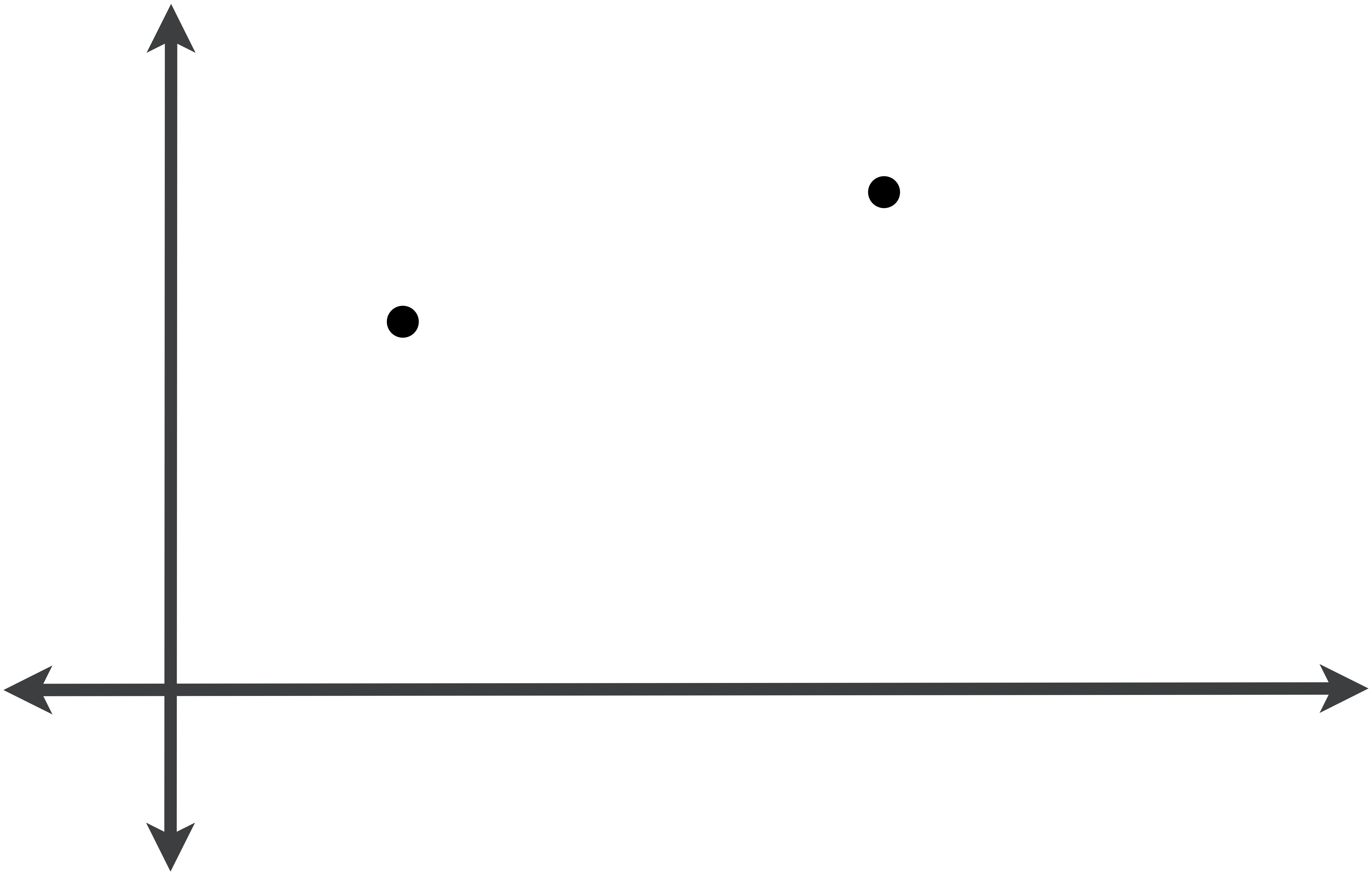
$$TE = mgh + \frac{1}{2}mv^2$$

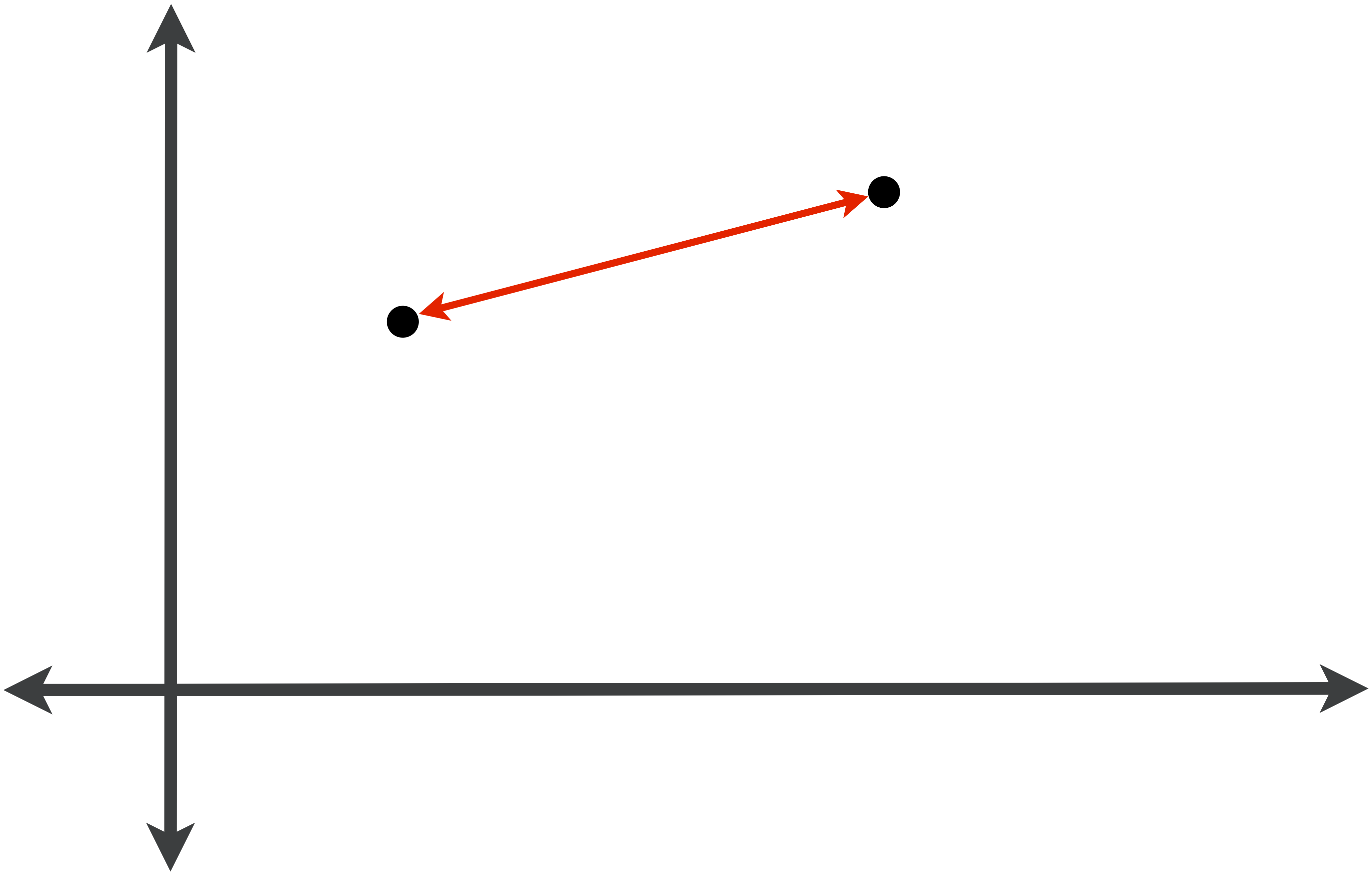


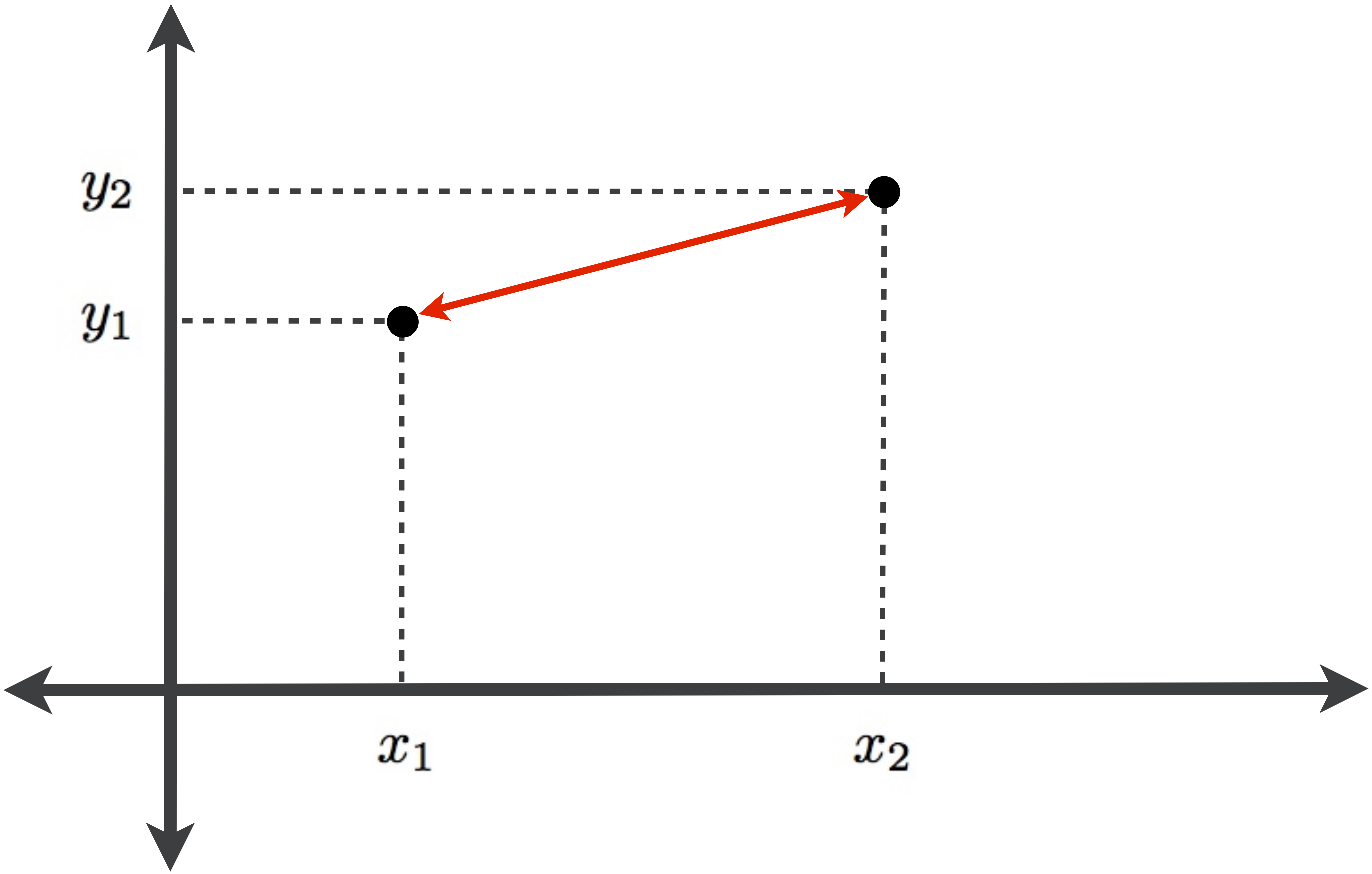
$$TE = mgh + \frac{1}{2}mv^2$$

$$TE = PE + KE$$

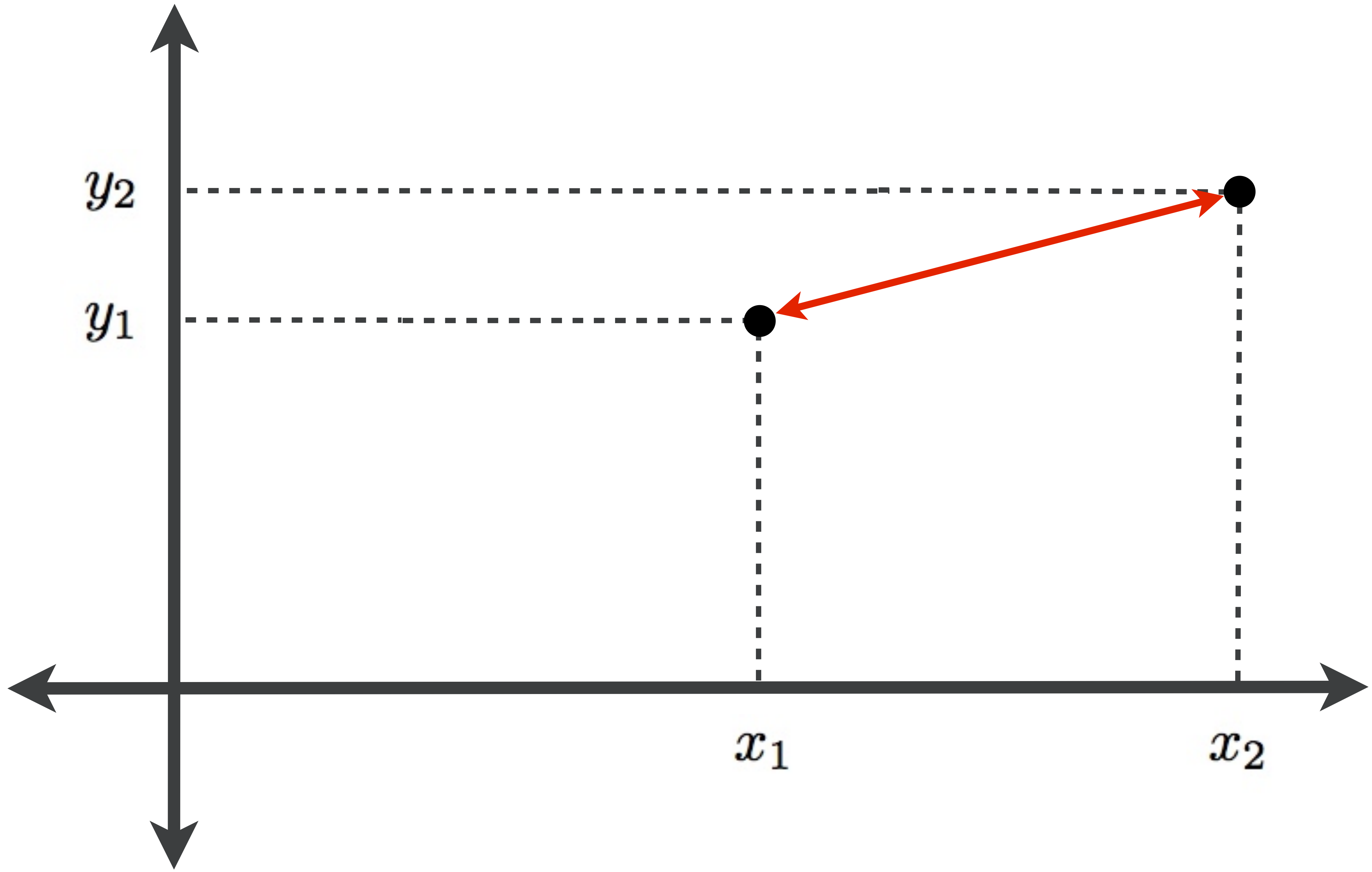








$$distance = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$



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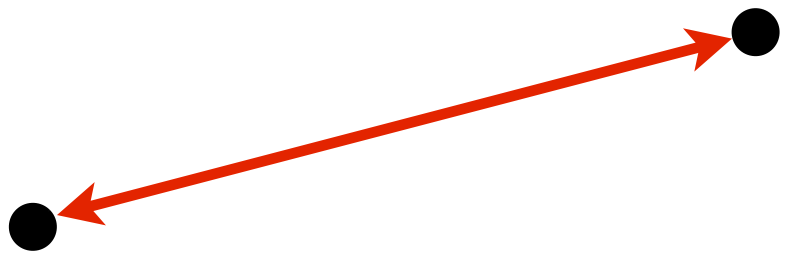
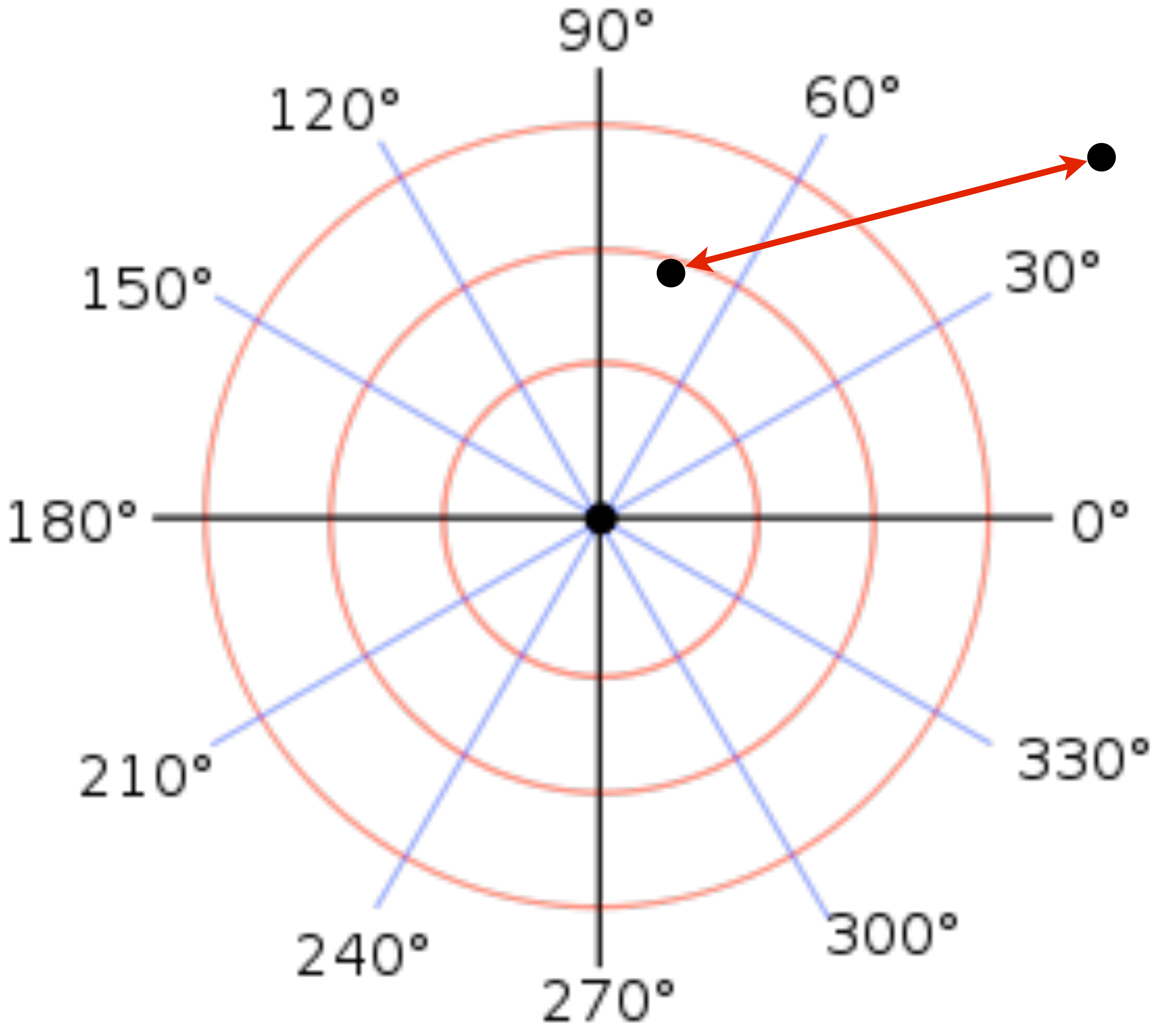
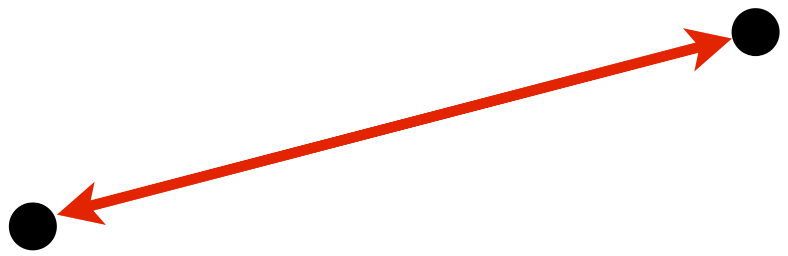
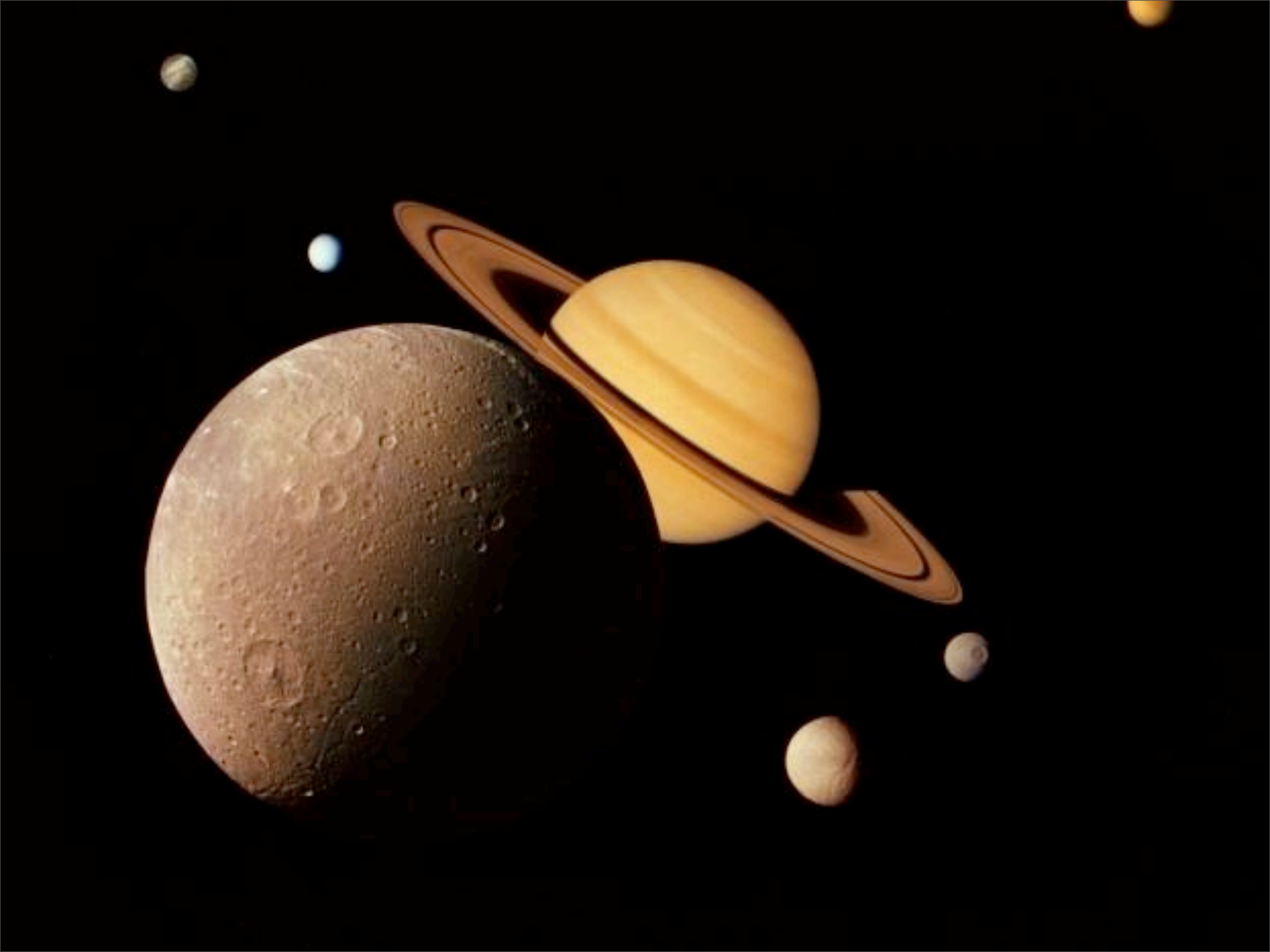
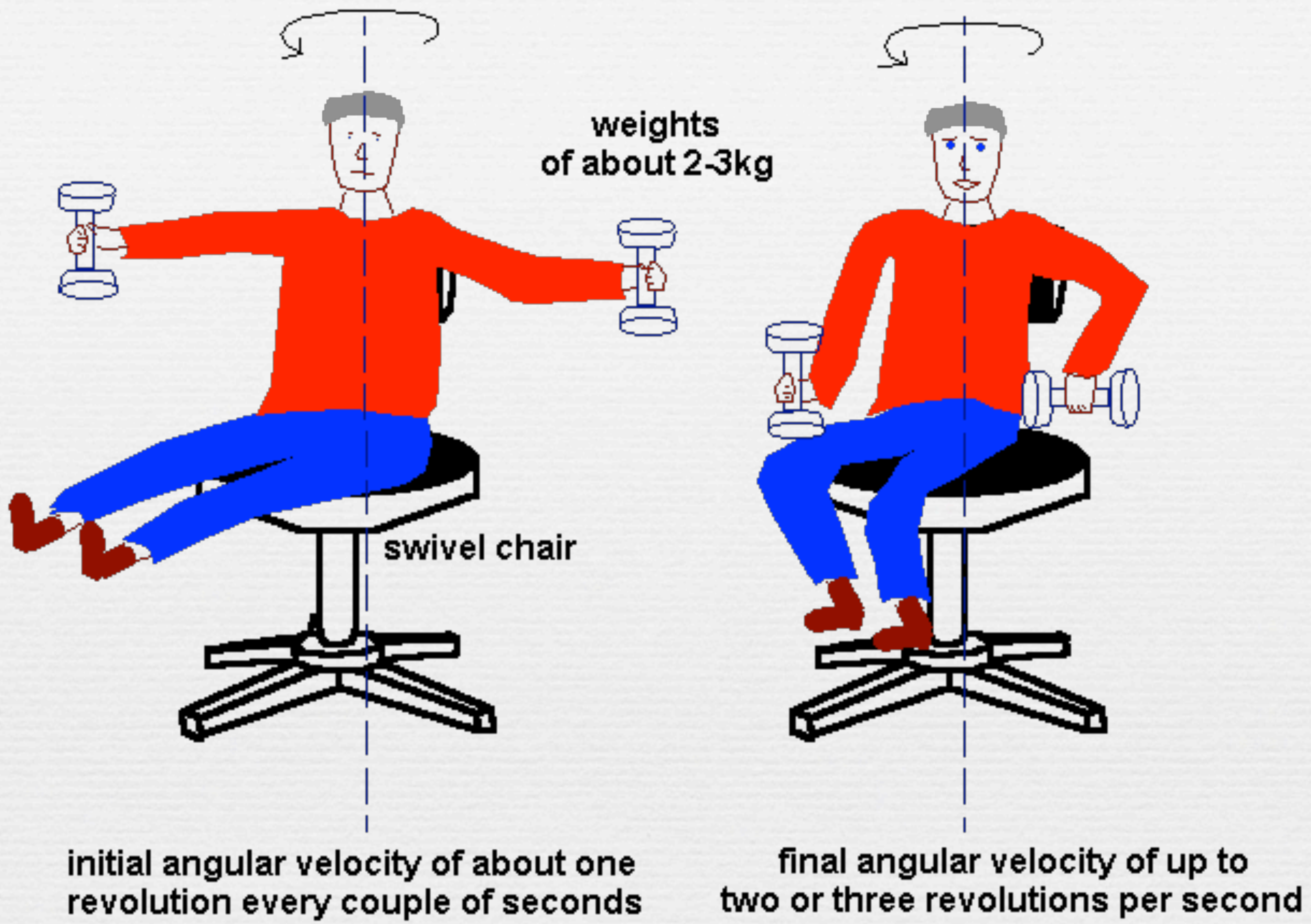



90°
120°
60°
150°
30°
180°
0°
210°
330°
240°
300°
270°



Distance is invariant under translation and change of coordinate representation.



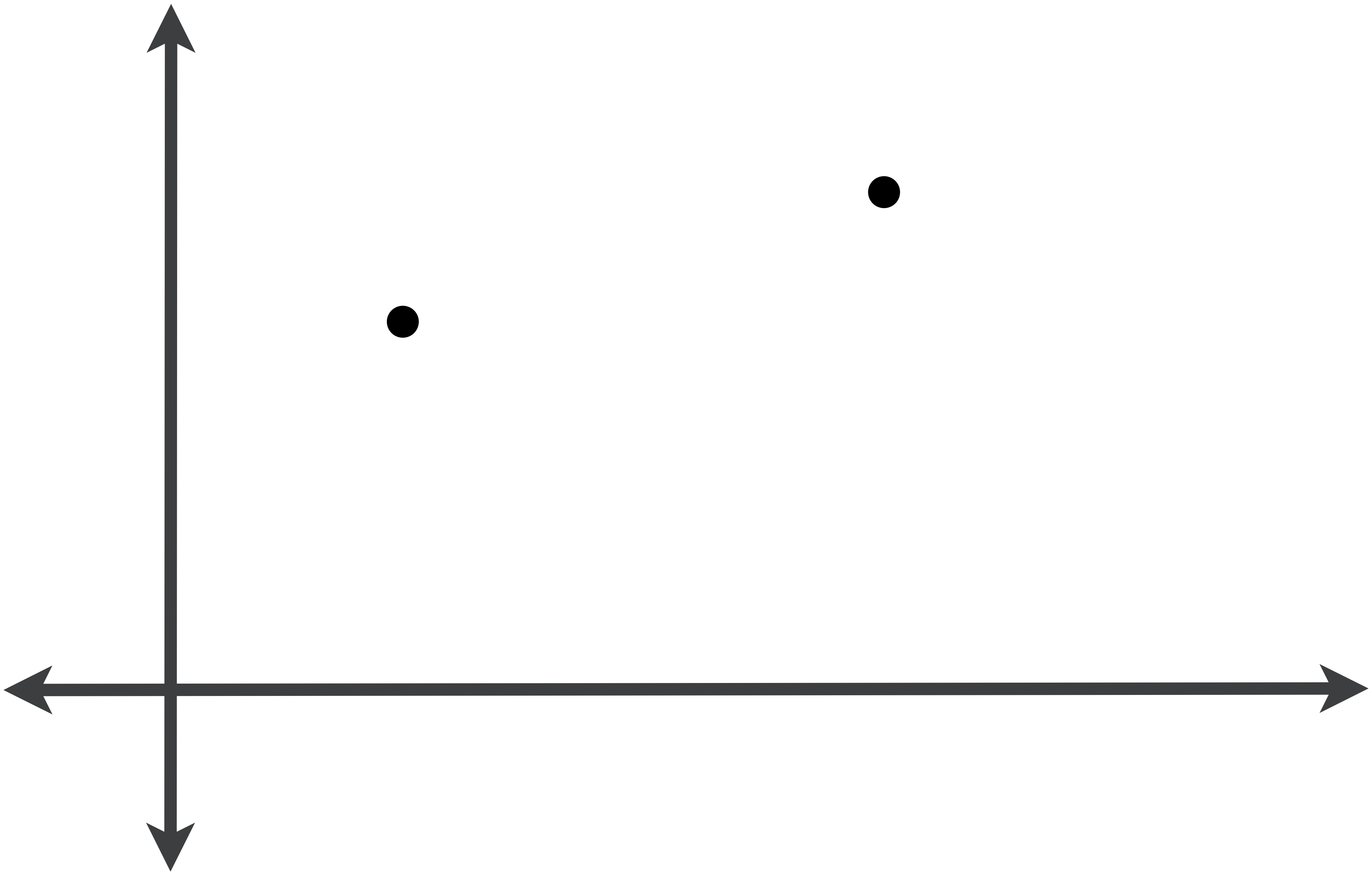

weights
of about 2-3kg
swivel chair
initial angular velocity of about one
revolution every couple of seconds
final angular velocity of up to
two or three revolutions per second

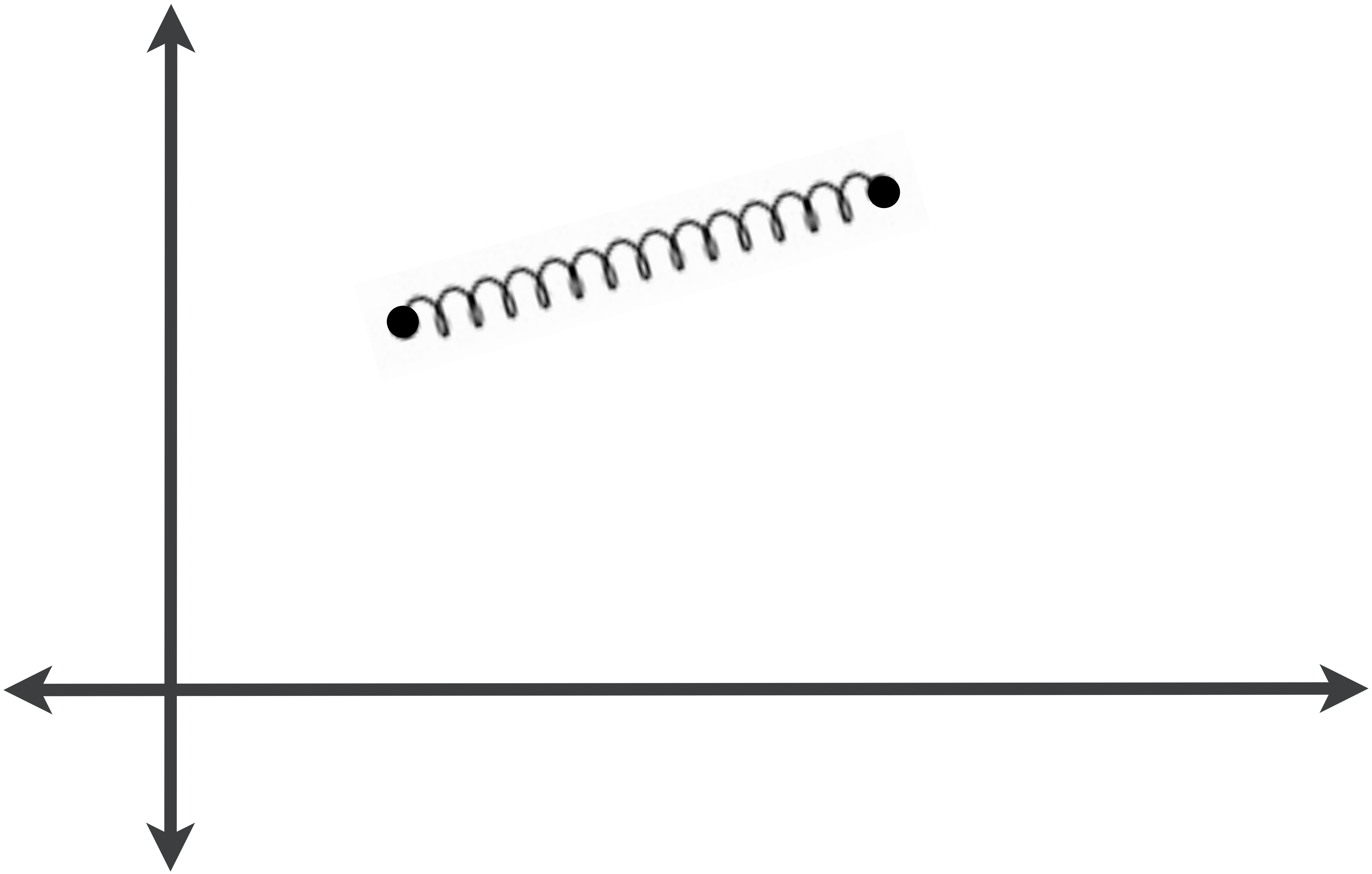
Noether's Theorem

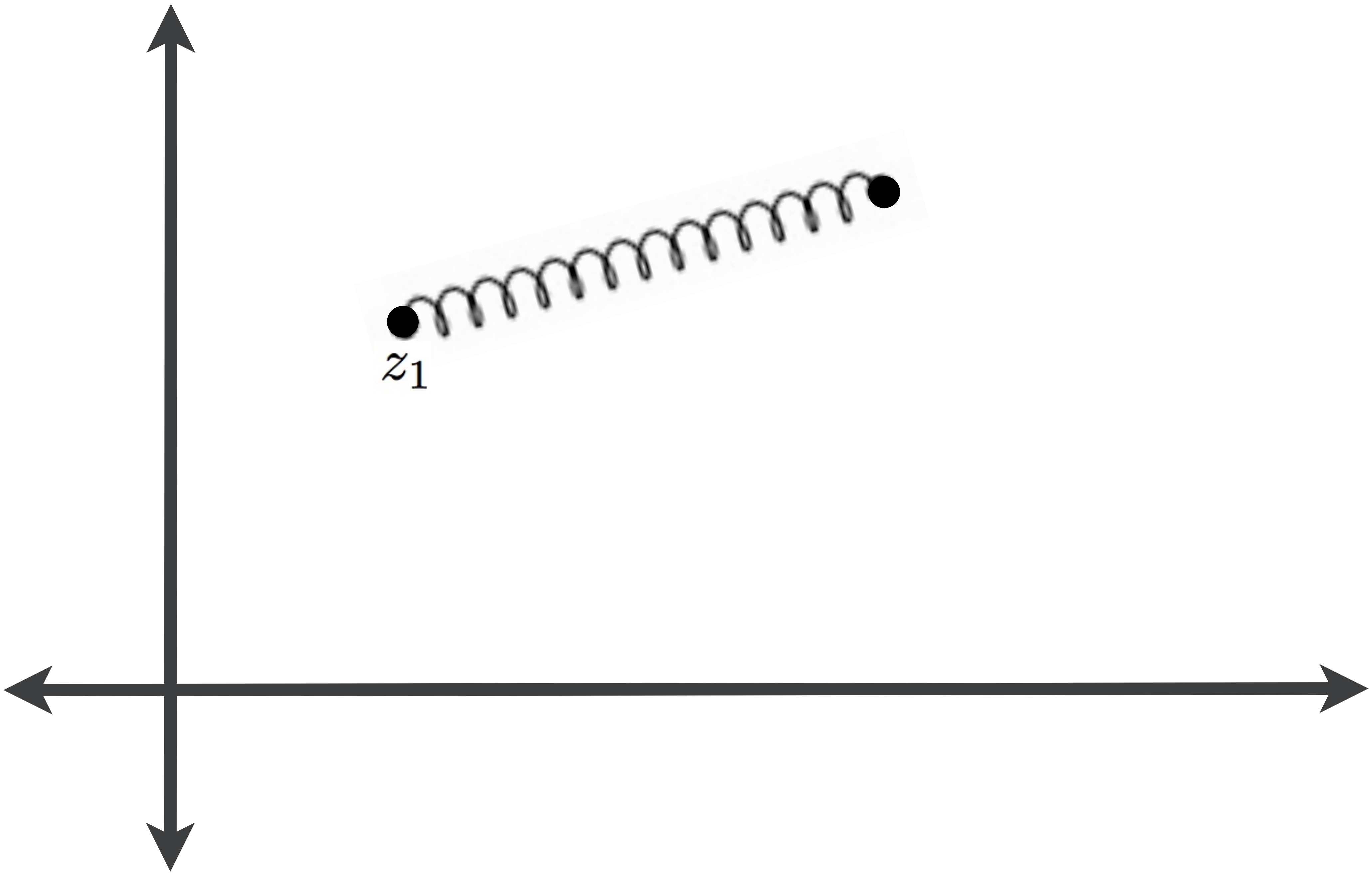
Noether's Theorem

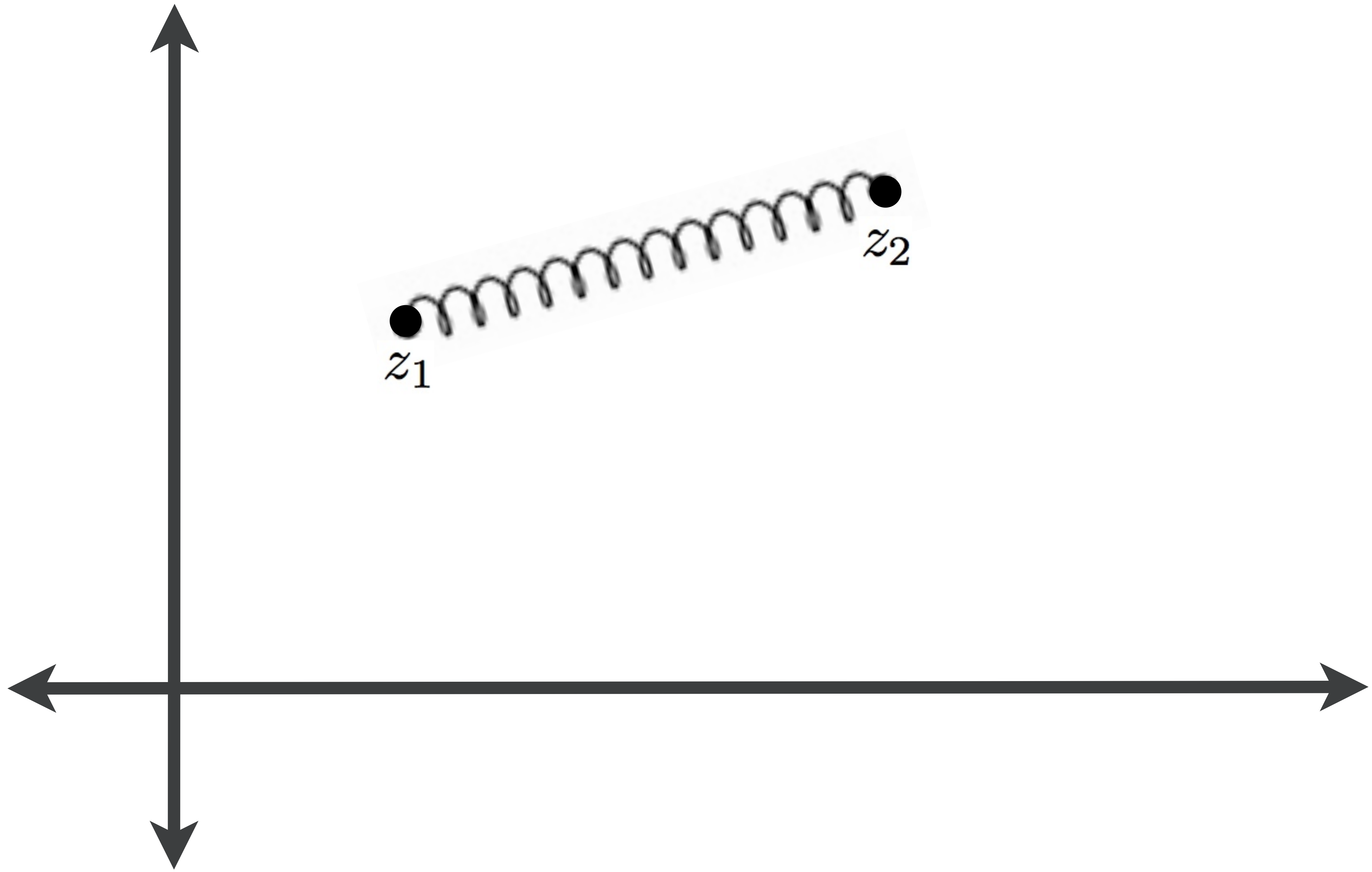
- (1915) "Any differentiable symmetry of the action of a physical system has a corresponding conservation law"

Quick Example

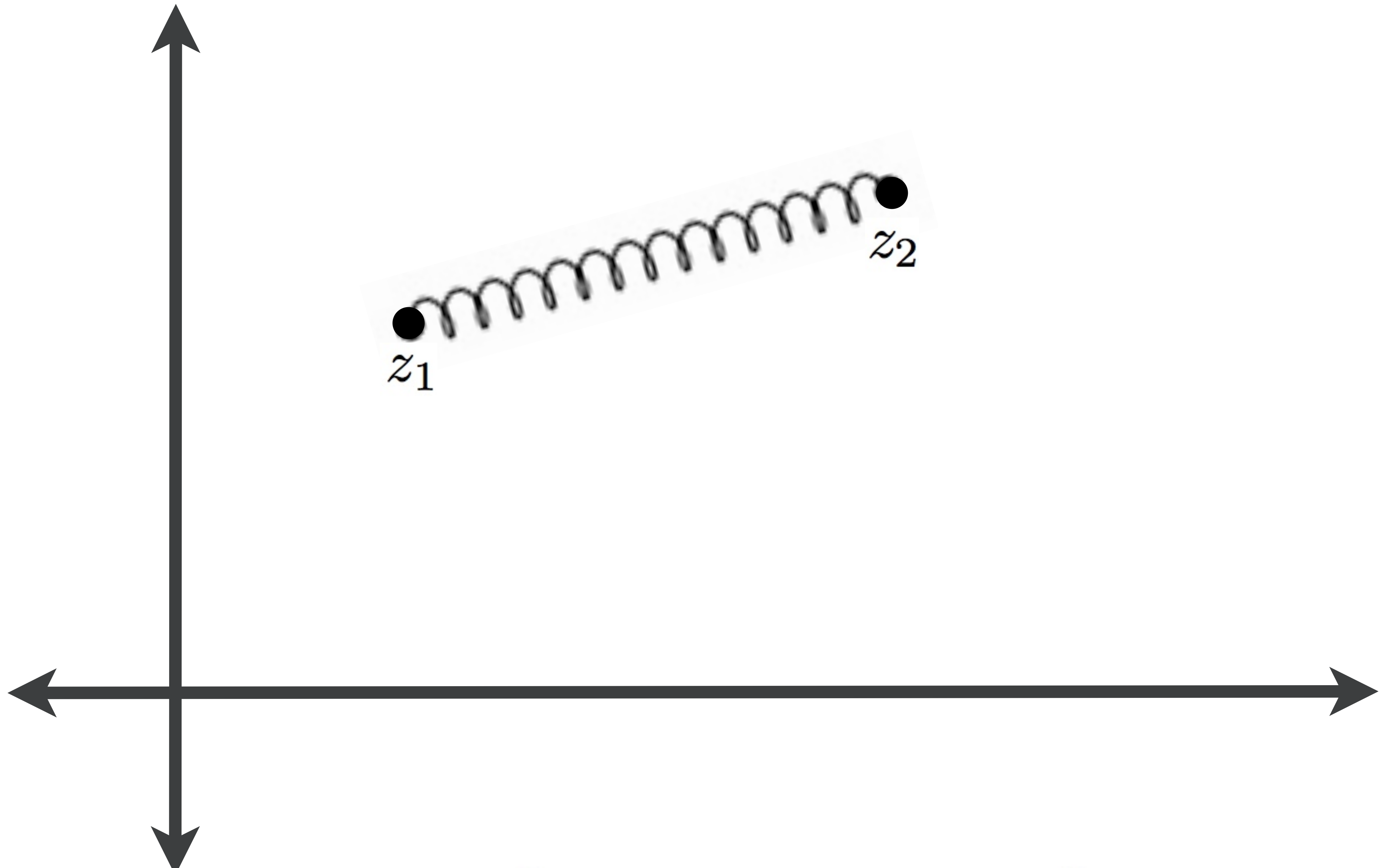





z_1



z_1
z_2



$$L(t, z_1, z_2, \dot{z_1}, \dot{z_2}) = \frac{1}{2}m(\dot{z_1}^2 + \dot{z_2}^2) - \frac{1}{2}k(z_1 - z_2)^2$$

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(Noether's Theorem)

$$\frac{d}{dt}m(\dot{x_1} + \dot{x_2}) = 0$$

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Conservation of Momentum

If the action

$$\mathcal{S}[q; a; b] = \int_a^b L(t, q, \dot{q})\, dt$$

is invariant under Φ_ϵ and Ψ_ϵ, then

$$\frac{d}{dt}\left(\sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\psi_i + \left(L - \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)\phi\right) = 0$$

where $\phi = \left.\frac{\partial \Phi}{\partial \epsilon}\right|_{\epsilon=0}$ and $\psi = \left.\frac{\partial \Psi}{\partial \epsilon}\right|_{\epsilon=0}$

Pretty cool, right?

TYPES

```
(λ x : unit . 42 : int)
```

$$\tau$$

```
π ≜ ΛT1 . ΛT2 . ΛT3 .
  λ v : T1 × T2 × T3 .
  v[T1] (λ x : T1 . λ y :
```

```
λ f : (int -> int) ref . λ n : int .
  f := (λ acc : int ref . λ m : int .
         case (n = m : bool) of
           (acc := (mul !acc  m); acc) : int
           (!f (acc := (mul !acc m); acc) (m+1)) : int
       ) (ref 1) 1) (ref λ x : int . x))
```

Atkey (2014)

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- Define a type system for Lagrangian Mechanics.

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- Derive conservation laws as "free theorems" by parametricity.

Lagrangian: $L(t, z_1, z_2, \dot{z}_1, \dot{z}_2) = \frac{1}{2}m(\dot{z}_1{}^2 + \dot{z}_2{}^2) - \frac{1}{2}k(z_1 - z_2)^2$

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Theorem (Noether). *Let $L(x, u, D_u^1, \ldots, D_u^n)$, be a Lagrangian for $A \subseteq \mathbb{R}^n$, let $\varphi \in Aut(A)$ be a symmetry of A such that*

$$\varphi(L) + LD^i(\xi) = D^i(B^i) \qquad B^i \in A$$

Then the Euler-Lagrange equations admit a conservation law $\forall i. D^i(C^i) = 0$.

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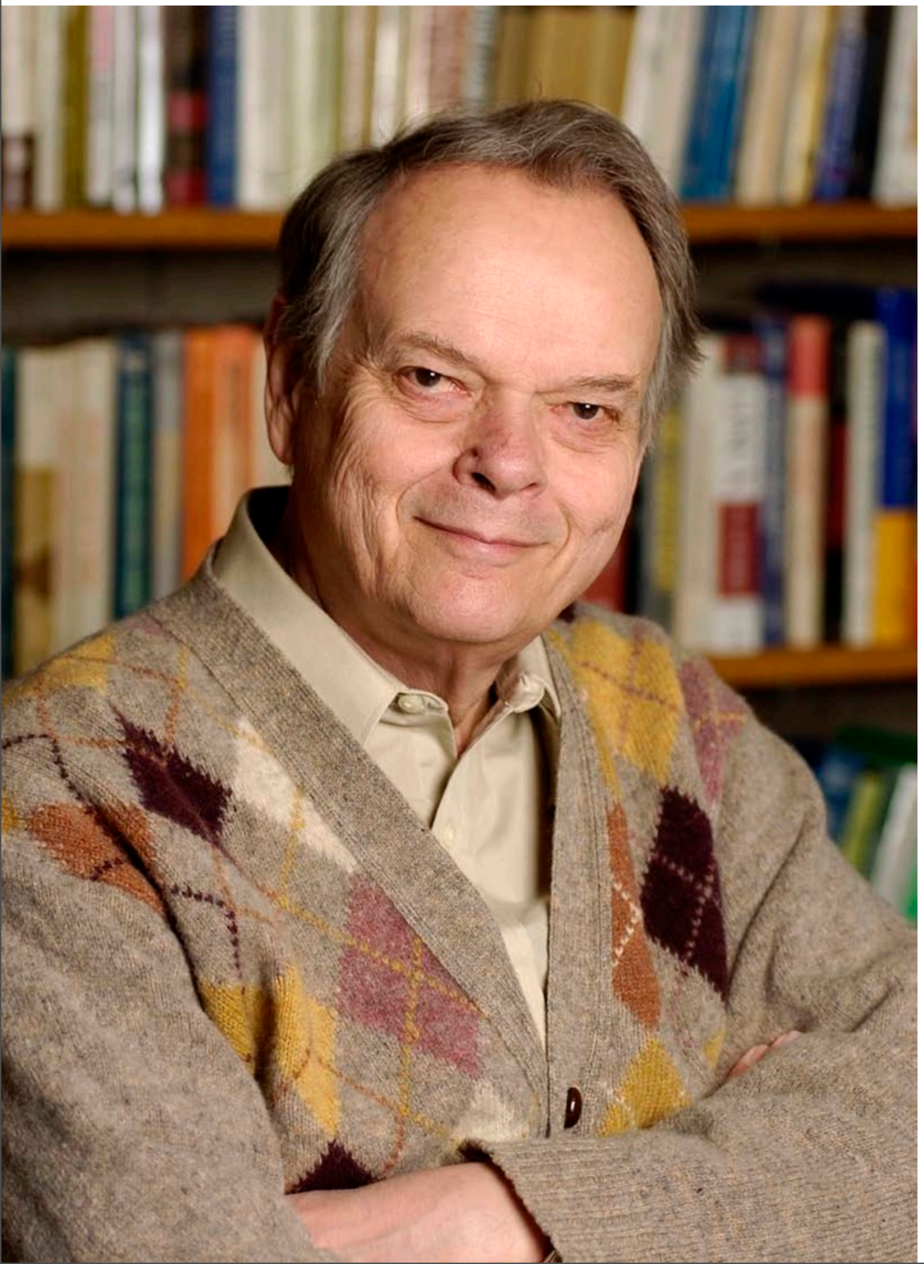
Key point: we need an automorphism (i.e. symmetry) to start with

What does this mean?

Reynolds:
types are relations



Reynolds:
types are relations

→

Wadler:
relations are free theorems

Reynolds:
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→

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Reynolds:
types are relations

→

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→

Atkey:
free theorems are symmetries

Atkey gives us a
geometric interpretation
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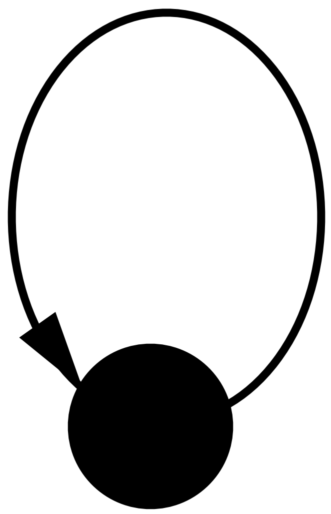
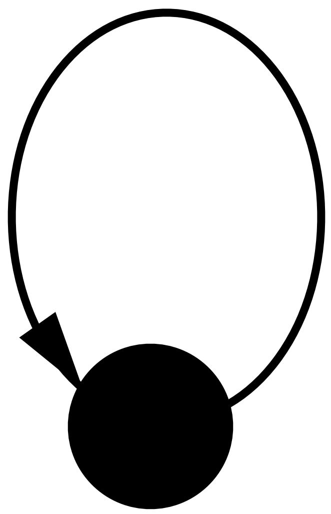
Atkey gives us a **geometric interpretation** of types

We'll argue: Atkey subsumes Reynolds + Wadler





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- Basic relation between Reynolds' types is the subset relation ($\subseteq$).
- Form a graph where the objects are types and the edges order types by $\subseteq$.

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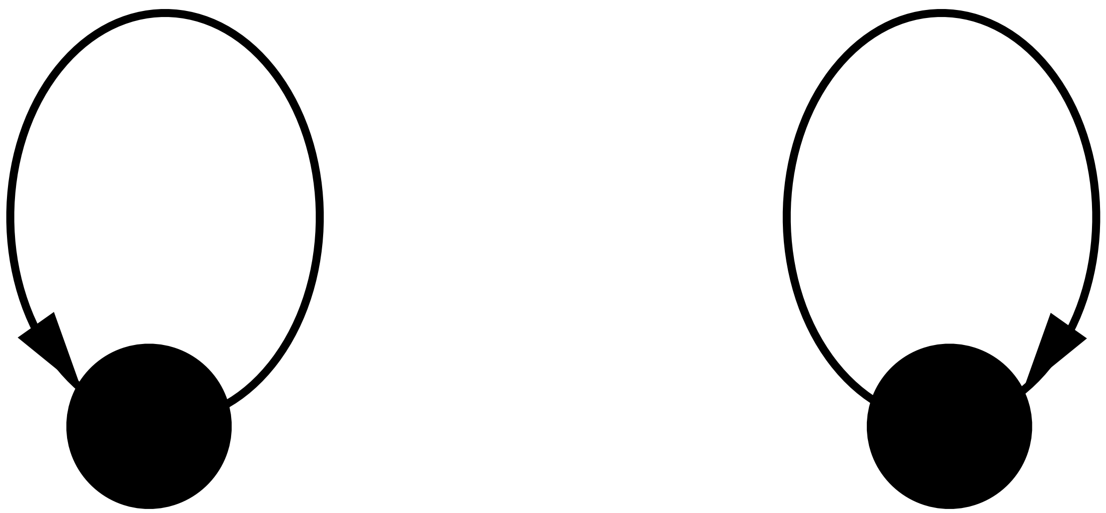

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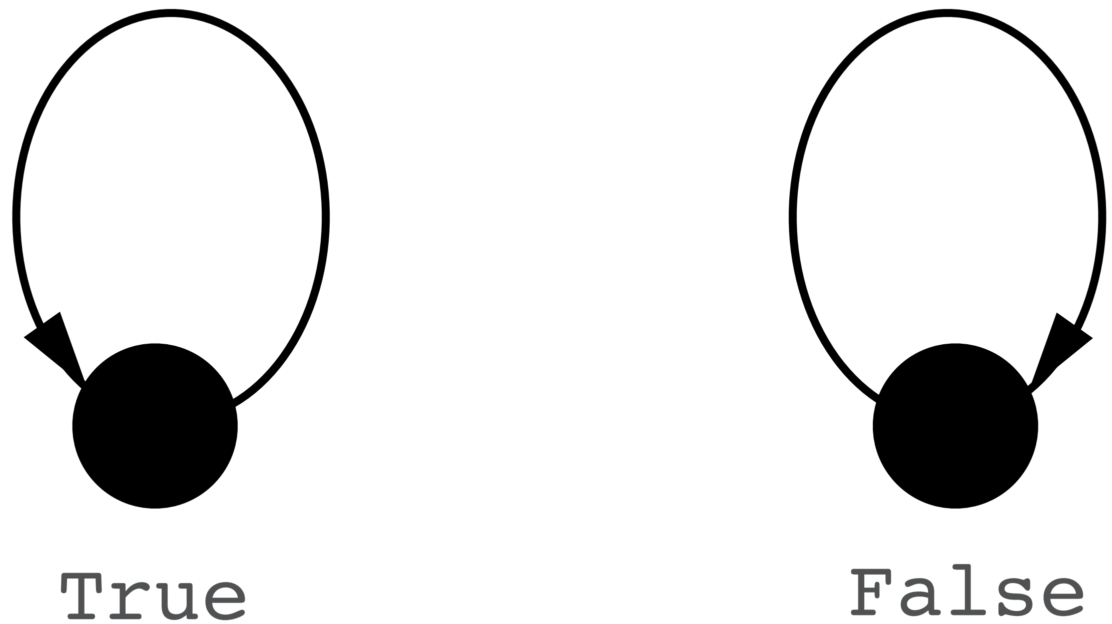

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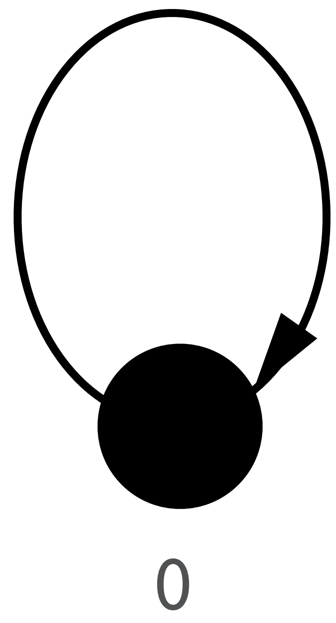

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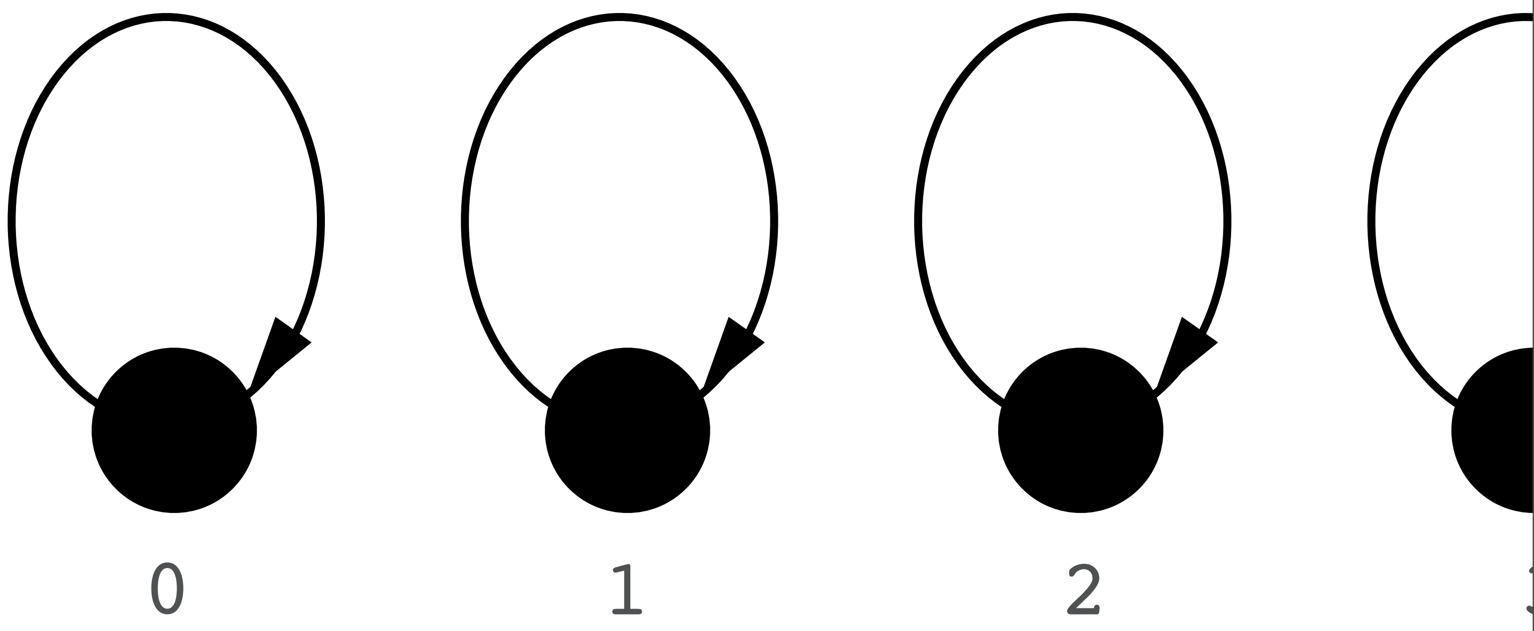

Example: cartesian space ($\mathbb{R}^1 \ldots \mathbb{R}^n$)

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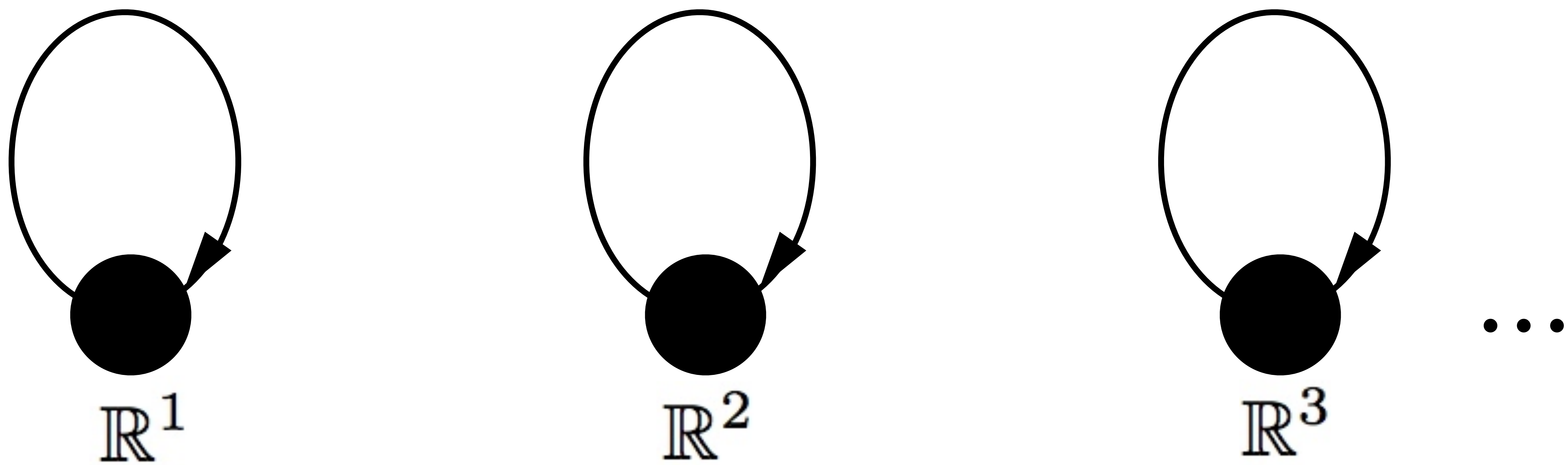

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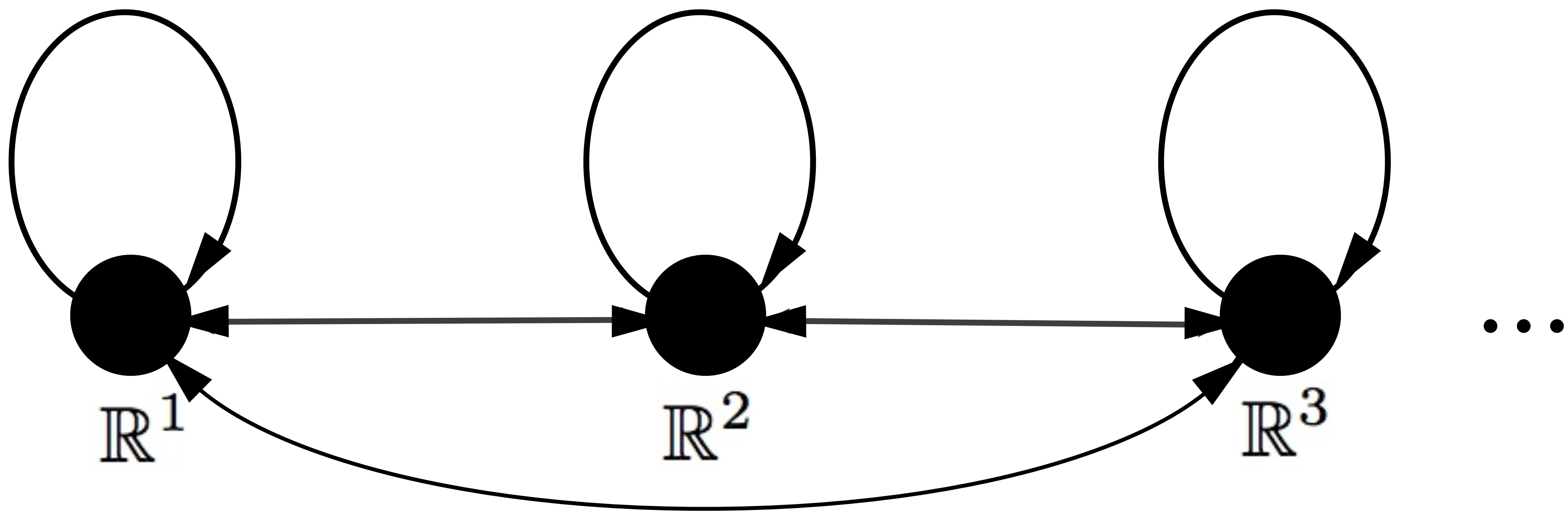

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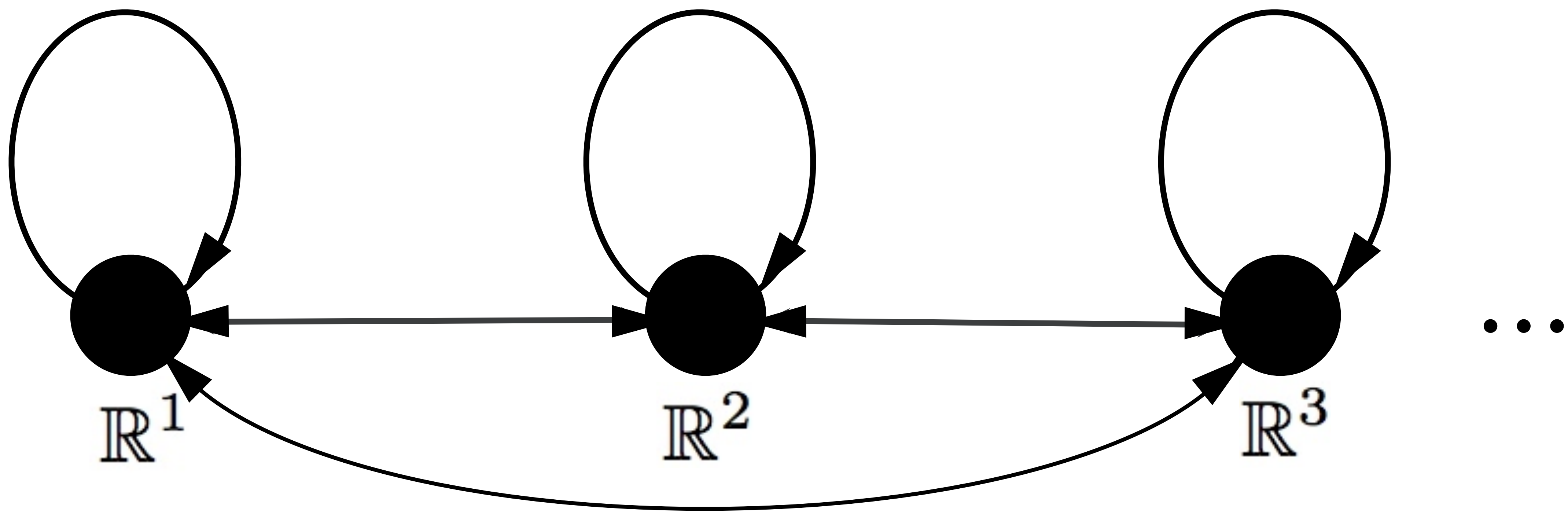

Each arrow represents a family of diffeomorphisms

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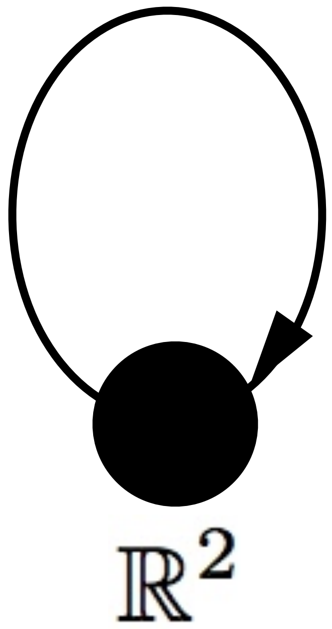

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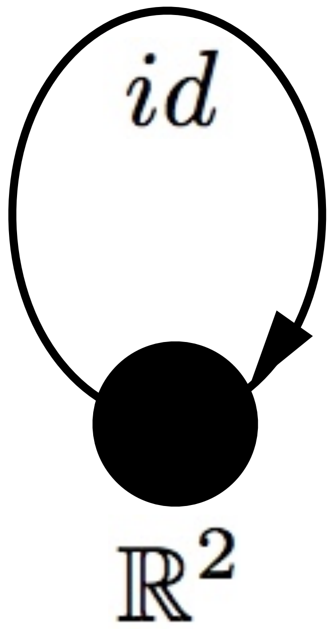

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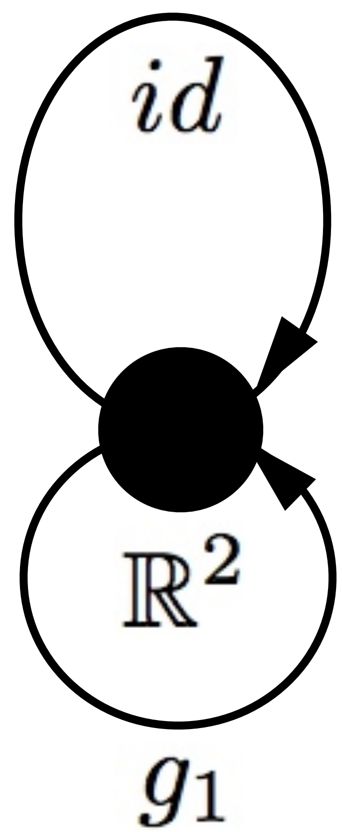

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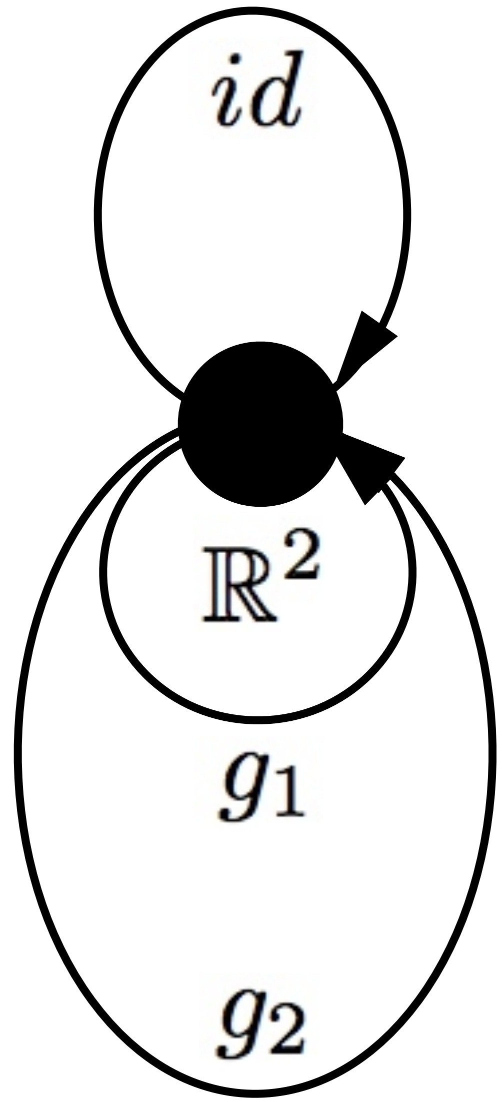

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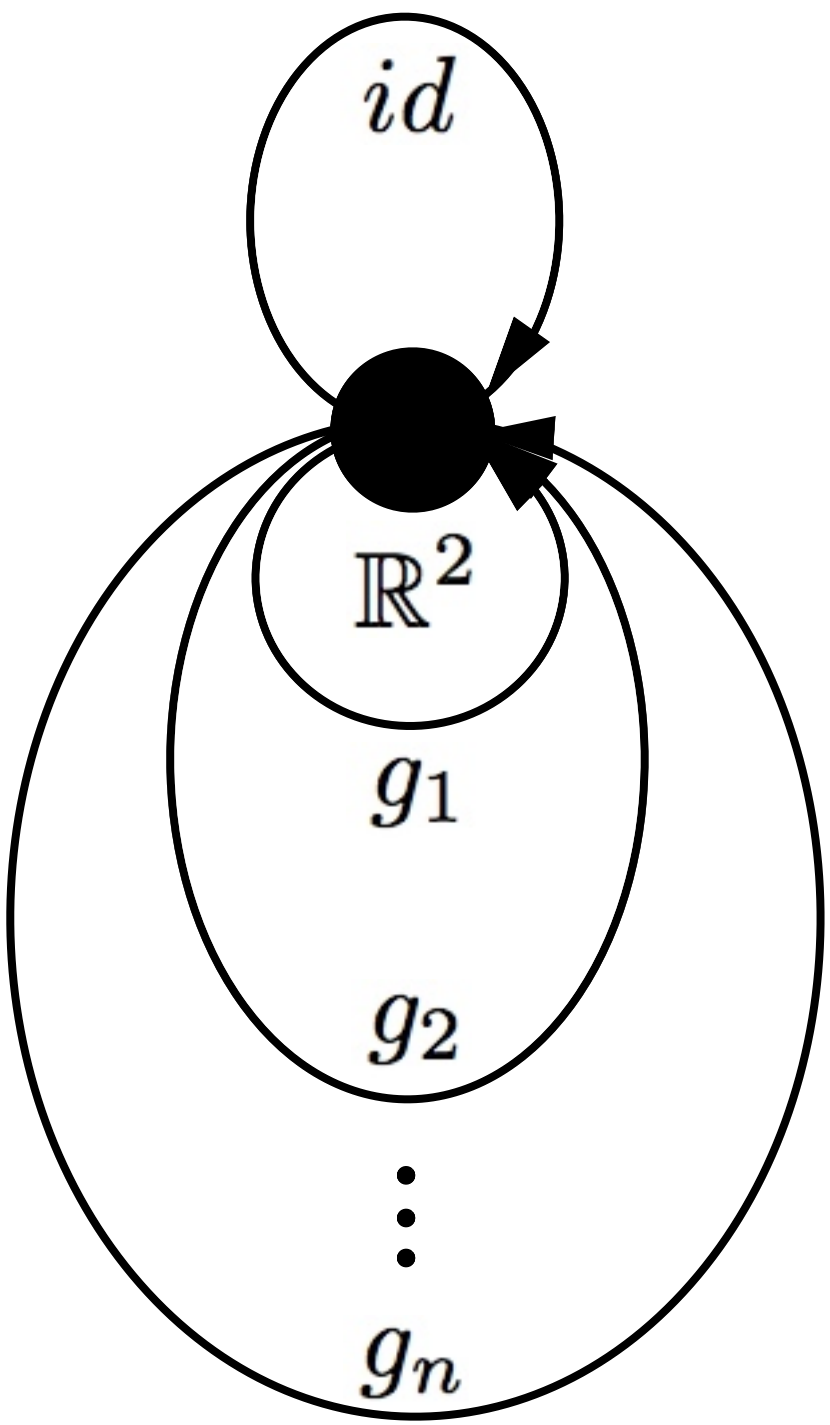

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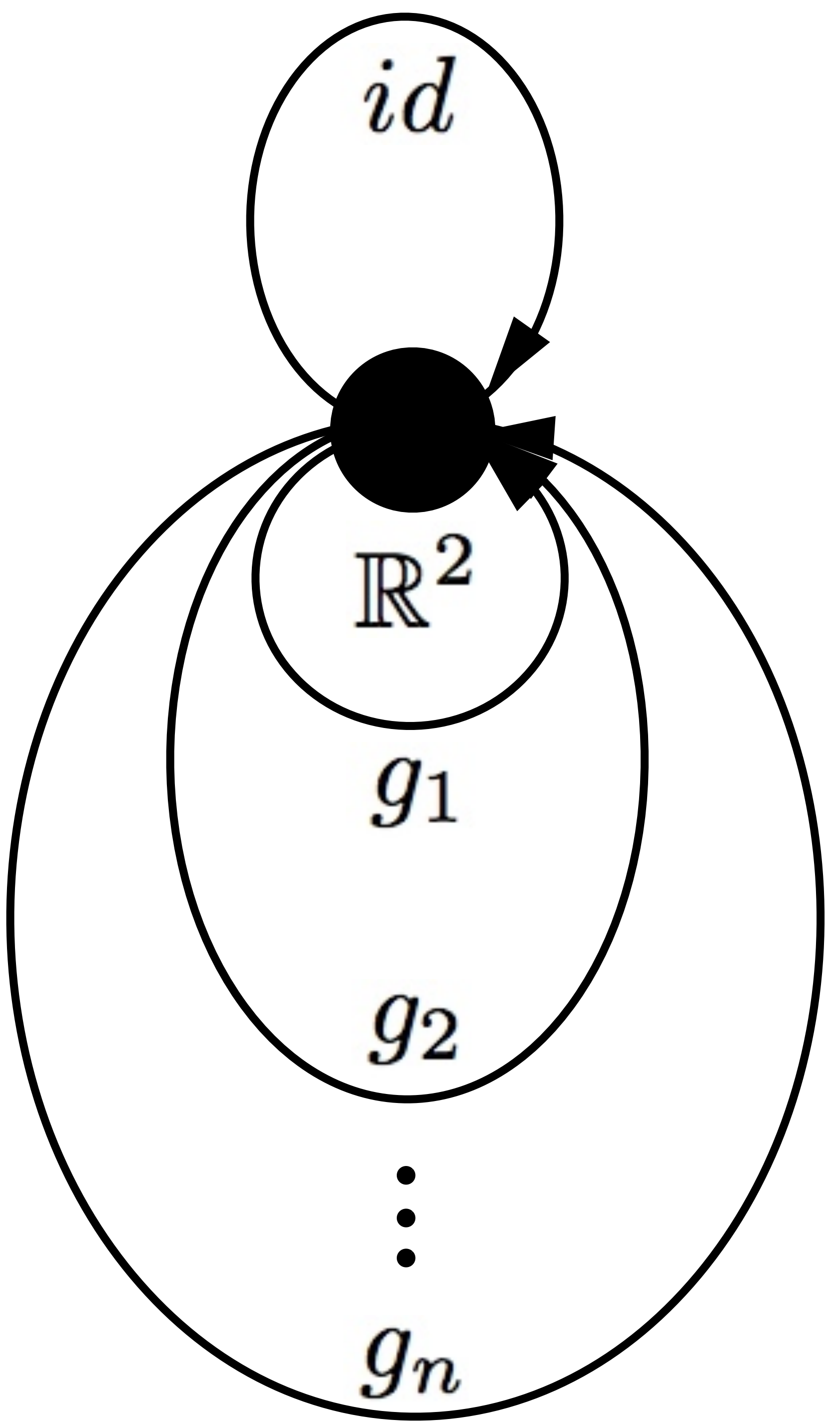

$g_i \in Aut(\mathbb{R}^2)$

Free Theorems

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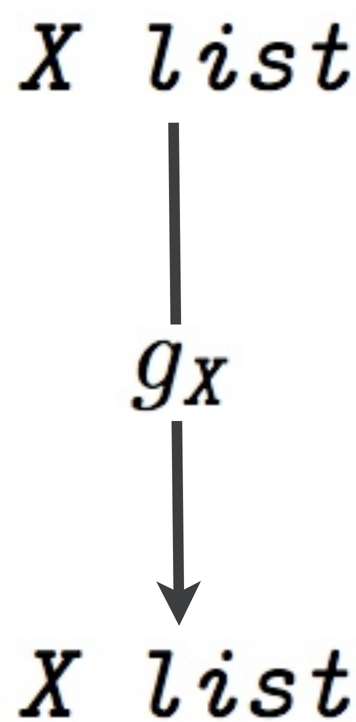

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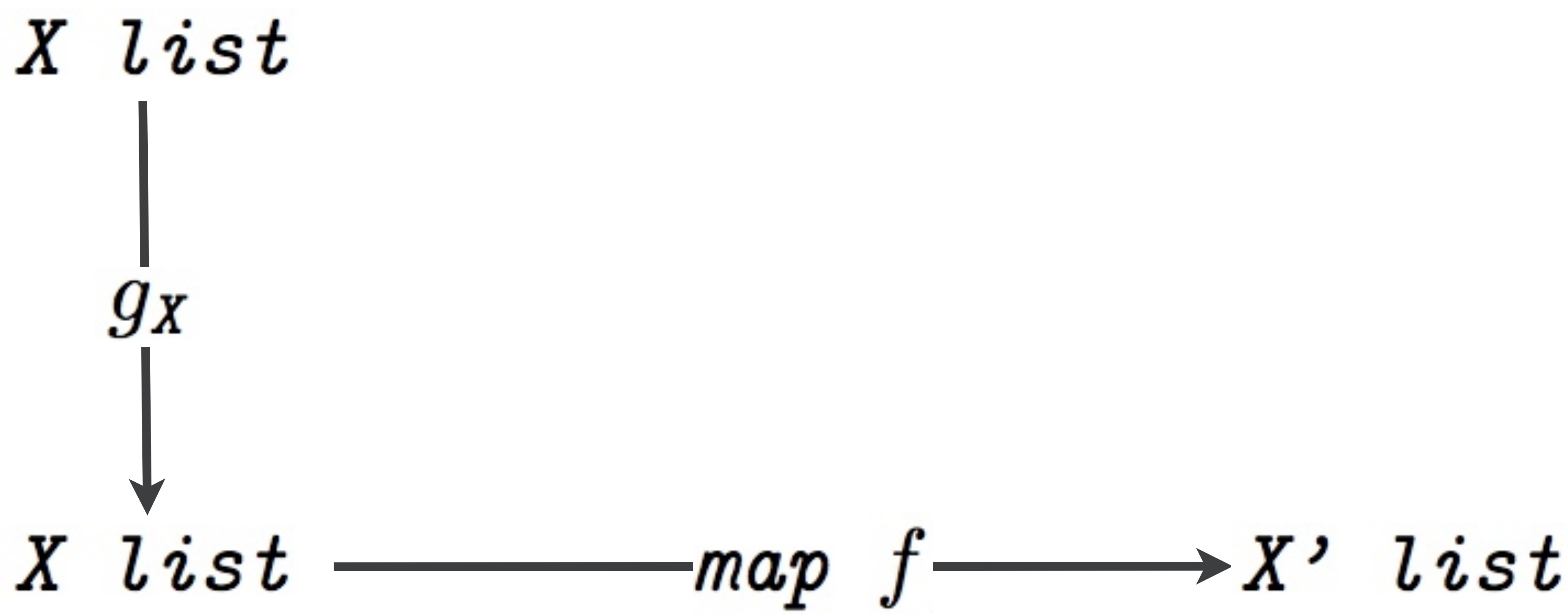

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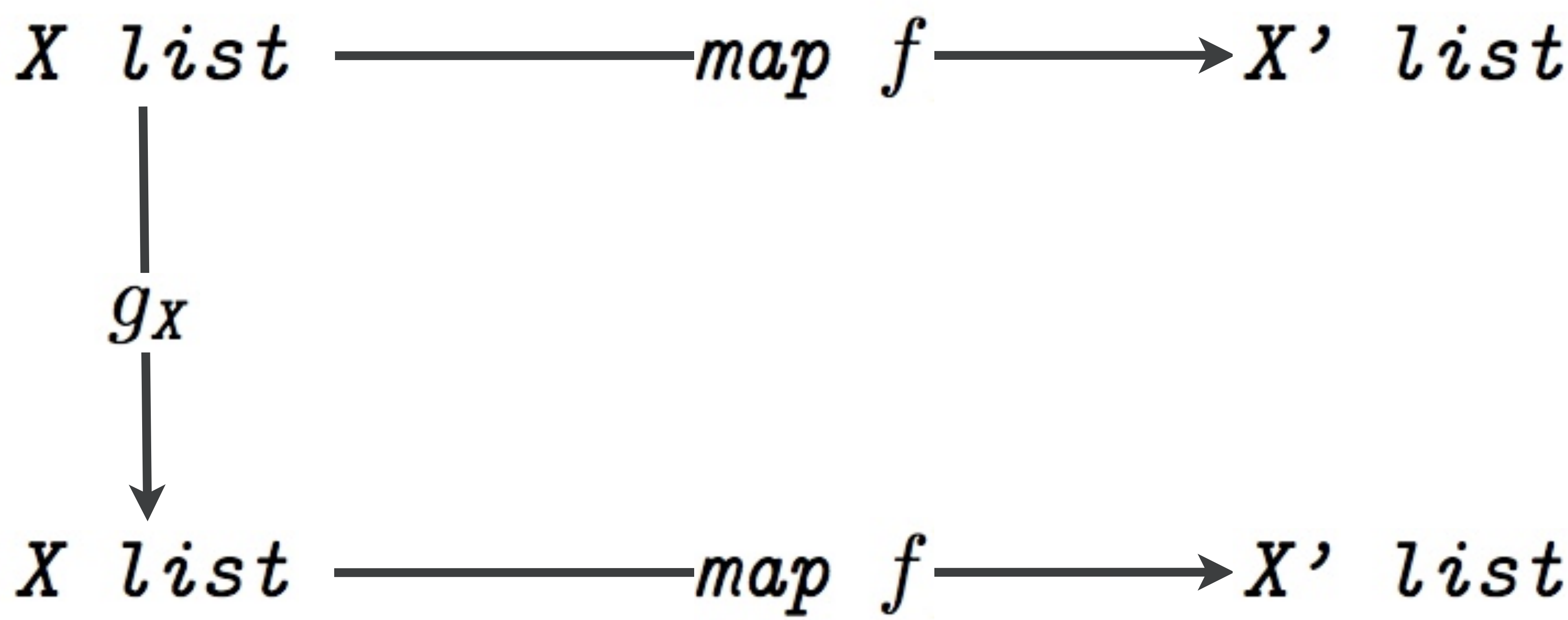

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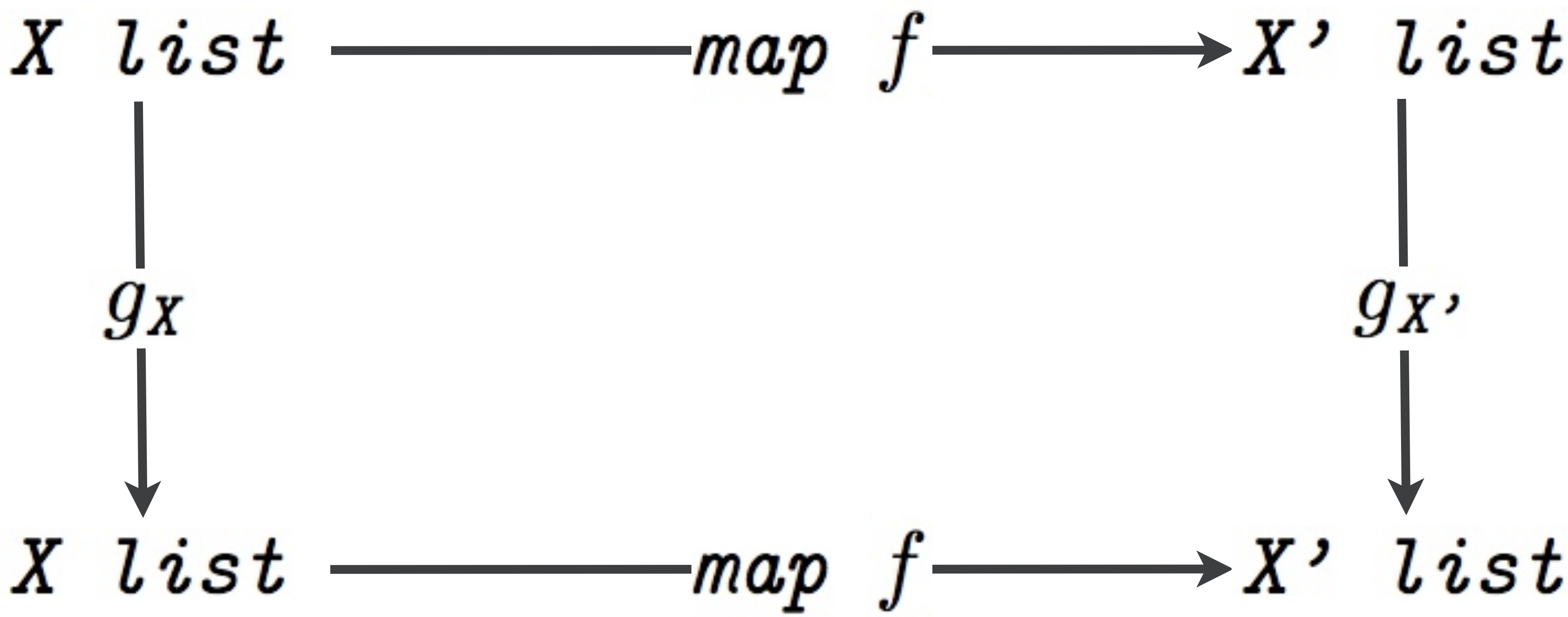

Atkey: Main Points

- Extend System Fω with type system encoding geometric invariances.
- Interpret kinds as reflexive graphs, types as reflexive graph morphisms.
- Connect free theorems of Wadler/Reynolds with Noether's theorem via symmetries of these reflexive graphs.

Atkey: Takeaways

- Types as geometries is a powerful new way of manipulating our "syntactic discipline".
- Visual intuition, connections to group theory.
- Physics is only one potential application!

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The End