Vertex-Based Formulations of Irradiance from Polygonal Sources

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Abstract

Irradiance at a point on a receiver due to a uniformly emitting polygon luminaire, or equivalently, the differential area to polygon form factor, is of fundamental interest in computer graphics. An elegant closed-form expression attributed to Lambert, dating from the 18th century, is the most commonly used formula for the problem. This document provides several alternatives to Lambert's formula, all of which are summations on the vertices of the source polygon rather than the edges. A term in a vertex-based summation is a function of the vertex position and the local behavior of the incident edges. The summations may be evaluated in any order and are therefore suited to algorithms where polygon contours are constructed incrementally.

1 Introduction

The fundamental radiometric quantity is *radiance*, the radiant power carried along a line [2]. Radiance is measured in units of W/m^2/sr. A related quantity is *irradiance*, the incident radiant flux at a surface point. Irradiance at a receiver point r is computed by integrating the incoming radiance against a cosine, in all directions above the surface:

$$I(r) = \frac{1}{\pi}\int_{\Omega} L(r,\omega)\cos(\theta)\,d\omega \tag{1}$$

where $L(r,\omega)$ is the incoming radiance at r in the direction ω and θ is the angle ω makes with the surface normal at r. Irradiance is the power per unit area at r, and thus carries the units W/m^2. Note that irradiance is defined at a point on a surface where there is a tangent plane and a well-defined outward normal. The tangent plane at r is also referred to as the *receiver plane*.

Irradiance and its photometric analog *illuminance* are important quantities in a variety of different areas, from biology to illumination engineering. For computer graphics, irradiance is of fundamental interest because it is directly proportional to the apparent brightness of a diffuse surface at a point.

1.1 Lambert's Formula

The irradiance due to a uniformly emitting polygon P can be computed by an elegant formula attributed to Lambert [9]

$$I(r) = \frac{M}{2\pi}\sum_{i=1}^{n}\beta_i\cos\gamma_i \tag{2}$$

where β_i is the angle edge i makes with the receiver point r, and γ_i is the angle the plane containing edge i and r makes with the receiver surface normal. M is an emission constant (in W/m^2). For a real surface, it is assumed that no portion of the polygon lies below (with respect to the surface normal) the tangent plane at r, and r is off the polygon.

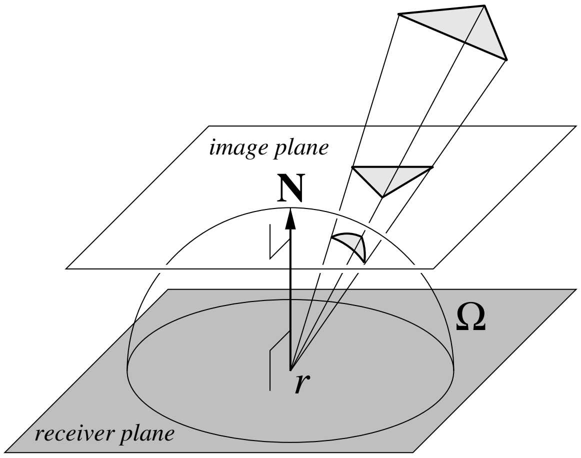


Figure 1: *A uniformly emitting polygonal source can be projected onto an image plane or the unit sphere through a receiver point r. The irradiance at r does not change, and can be computed from a surface integral on any of the three polygons.*

Lambert's formula may be expressed directly in terms of the vertices, $v_1, \ldots v_n$ of the polygon, the receiver point r, and the unit surface normal **N** at r:

$$I(r) = -\frac{M}{\pi} \sum_{i=1}^{n} \arccos \frac{\mathbf{v}_i \cdot \mathbf{v}_{i+1}}{\|\mathbf{v}_i\| \|\mathbf{v}_{i+1}\|} \frac{\mathbf{v}_i \times \mathbf{v}_{i+1}}{\|\mathbf{v}_i \times \mathbf{v}_{i+1}\|} \cdot \mathbf{N}. \tag{3}$$

where $\mathbf{v}_i = v_i - r$. In this formulation, the edge normals $\mathbf{v}_i \times \mathbf{v}_{i+1}$ are the outward normals of the polyhedral cone through the polygon with apex at r, hence the negative sign preceding the sum.

A drawback to Lambert's Formula is that it is a summation over the edges of the polygon rather than the vertices. For an unoccluded polygon this is less of a consideration, but algorithms for computing partially occluded irradiance often work by clipping the source polygon against all the occluding polygons [7]. This process generally produces the vertices of the clipped source before it produces the edges, so a formula for irradiance that is a summation over the vertices of the polygon rather than the edges is a useful alternative. The development of such vertex-based formulas is the purpose of this work.

1.2 Plan of this Work

An important property of uniformly emitting objects is illustrated in Figure 1. Projectively equivalent uniformly emitting objects produce the same irradiance; that is, objects which look the same from a given receiver point produce the same irradiance. This follows directly from (1), which is an integral over only the incoming directions. The advantage of this property is that projecting a uniformly emitting source polygon onto an image plane, or onto the hemisphere, will produce the same irradiance.

This document develops formulas for the irradiance due to a uniformly emitting polygon based on the vertex positions and the local behavior of the edges at the vertices. In Section 2, the polygon is projected onto an image plane and a formula is developed in terms of the projected vertices and the slopes of the incident edges. Section 3 details a more direct reformulation of Lambert's formula which works on the original polygon vertices and uses edge vectors and normals for local behavior at each vertex. In Section 4 a formula is developed based on a spherical projection of the polygon, and in Section 6 the image plane formula is extended to work in the real projective plane.

Whenever possible, the formulas will be developed in terms of a natural representation of the polygon in the space in which the formula is developed rather than referring to the original polygon vertices.

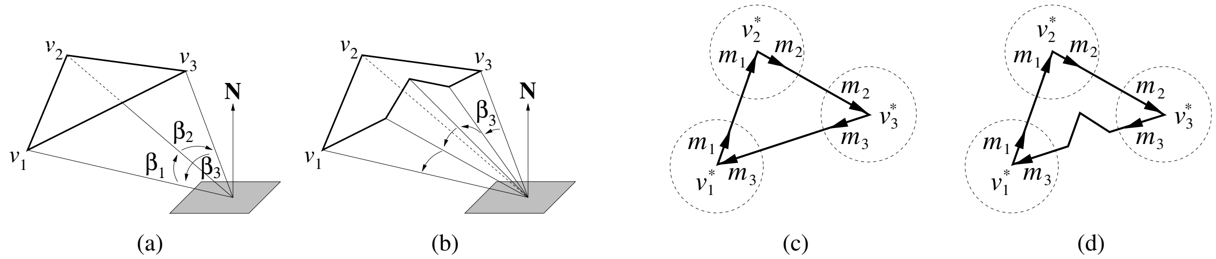


Figure 2: (a) *The geometry for Lambert's formula.* (b) *The angles β_i depend on the edges, so if part of the polygon is clipped by an occluder, the terms associated with the vertices of the affected edges have different values.* (c) *Using Green's theorem in the image plane produces a formula in terms of the local behavior at the vertices.* (d) *The contributions of the existing vertices are not affected if a bite is taken out of the polygon.*

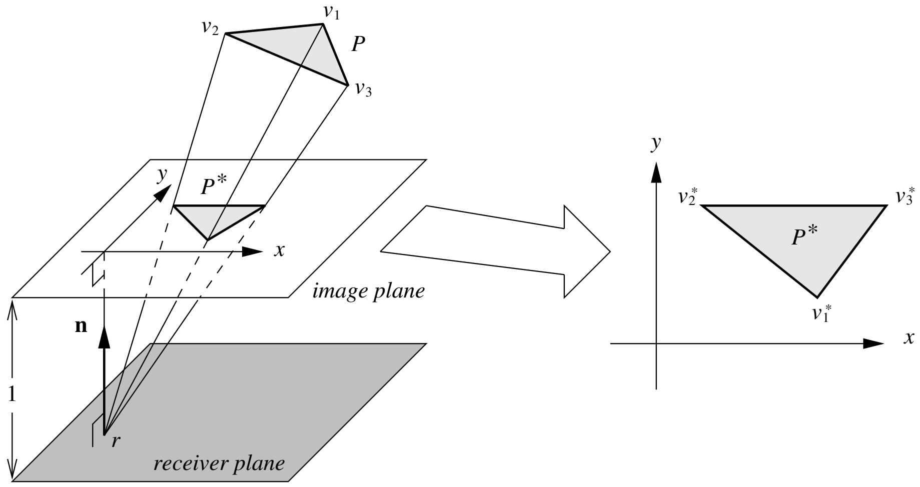


Figure 3: *To apply Green's theorem, the polygon P is projected onto an image plane parallel to the receiver plane, one unit above. The origin of the coordinate system of the image plane lies directly above the point of evaluation r on the receiver. The projection induces a reversal of orientation for front-facing polygons.*

2 Image Plane Formula

The objective of this section is to construct a formula in terms of the vertices of a polygon P that has been projected through r onto an *image plane*, which is the plane parallel to the surface at r and one unit above (in the direction of the outward normal $\mathbf{n}$ at r) as shown in Figure 3. This projection does not change the irradiance at r [2]. The behavior of the edges incident on each vertex may then be characterized by their slopes.

Lambert's formula shows that the irradiance is invariant under rotation about the normal $\mathbf{n}$, so the orientation of the x and y-axes in the image plane is not important. If $\mathbf{u}$ is an arbitrary unit vector perpendicular to $\mathbf{n}$, and $\mathbf{v} = \mathbf{n} \times \mathbf{u}$, the projection of a vertex v of P may be computed, for example, using the homogeneous transformation

$$v^* = \left[\begin{array}{cccc} & \mathbf{I} & & \mathbf{0} \\ 0 & 0 & 1 & 0 \end{array} \right] \left[\begin{array}{cccc} \mathbf{u} & \mathbf{v} & \mathbf{n} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{array} \right]^T \left[\begin{array}{cc} \mathbf{I} & -r \\ \mathbf{0}^T & 1 \end{array} \right] \left[\begin{array}{c} v \\ 1 \end{array} \right]. \tag{4}$$

(In the case of a polygonal receiver, $\mathbf{u}$ can be a normalized edge; for a curved receiver, $\mathbf{u}$ could be the direction of one of the curvilinear coordinates.) The first matrix on the left effects the projective projection, the other two perform a rotation and translation, respectively, to a coordinate system in which the receiver plane is the xy-plane and the image plane is the parallel plane through $(0, 0, 1)$, coordinates in this system will be called *image plane coordinates*.

In the remainder of this section we shall assume P has been projected onto the image plane forming a new planar polygon P^* having vertices $v_1^*, \ldots, v_n^*$. Each vertex v_i^* of P^* will be treated as a two-dimensional point (x_i, y_i) in image plane coordinates; the z-coordinate is always 1 and is omitted in this section.

2.1 Integration

The irradiance from a uniformly emitting surface $\mathcal{S}$, which is not self-occluding as viewed from a point r on a receiver can be computed [1] from the surface integral

$$I(r) = \frac{M}{\pi} \int_{\mathcal{S}} \frac{\cos\theta_{\mathcal{S}} \cos\theta_r}{d^2}\, dS, \tag{5}$$

where d is the distance from r to a point on $\mathcal{S}$, θ_r and $\theta_{\mathcal{S}}$ are the angles made by the ray joining r and the point with the receiver normal at r and the surface normal at the point, respectively. The constant M is an emission constant of $\mathcal{S}$.

For the projected polygon P^*, the integral of (5) has a particularly simple form; it reduces to the ordinary plane double integral (omitting the emission constant and normalizing factor $1/\pi$)

$$I(r) = \int\!\!\int_{P^*} \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy. \tag{6}$$

This double integral may be reduced to a contour integral on the boundary of P^* using Green's theorem:

$$\oint_{\partial P^*} F_1\, dx + F_2\, dy = \int\!\!\int_{P^*} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy. \tag{7}$$

The usual convention is counter-clockwise vertex ordering with respect to the outward normal. For a "front-facing" polygon, the angle between the outward normal and the receiver surface normal is negative, so the projected polygon P^* will have a clockwise vertex ordering on the image plane, which means a negatively-oriented boundary contour and the sign of the left-hand side of (7) must be reversed.

Taking $F_2(x, y) \equiv 0$ and $F_1(x, y)$ an anti-derivative of the integrand in (6) with respect to y we obtain from Green's theorem

$$\int\!\!\int_{P^*} \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy = \oint_{\partial P^*} F_1(x, y)\, dx = \sum_{i=1}^{n} \int_{E_i^*} F_1(x, y)\, dx.$$

The line integral over each edge can be evaluated by parameterizing the edge with the line equation $y = m_i x + b_i$ and integrating over the domain of the edge $E_i^* = \overline{v_i^* v_{i+1}^*}$

$$\int_{E_i^*} F_1(x, y)\, dx = \int_{x_i}^{x_{i+1}} F_1(x, m_i x + b_i)\, dx = \Omega(x_{i+1}, m_i, b_i) - \Omega(x_i, m_i, b_i)$$

(vertical edges consequently drop out of the summation). Here Ω is

$$\Omega(x, m, b) = \int \left(\int \frac{1}{(1 + x^2 + y^2)^2}\, dy \right) \Bigg|_{y = mx + b} dx,$$

$m_i = (y_{i+1} - y_i)/(x_{i+1} - x_i)$ is the slope of the segment joining v_i^* and v_{i+1}^*, and b_i is the y-intercept of that line.

The irradiance integral may therefore be written as

$$\begin{aligned} I &= \sum_{i=1}^{n} \Omega(x_{i+1}, m_i, b_i) - \Omega(x_i, m_i, b_i) \\ &= \Omega(x_2, m_1, b_1) - \Omega(x_1, m_1, b_1) + \cdots + \Omega(x_1, m_n, b_n) - \Omega(x_n, m_n, b_n) \\ &= \sum_{i=1}^{n} \Omega(x_i, m_{i-1}, b_{i-1}) - \Omega(x_i, m_i, b_i) \end{aligned}$$

As $b_i = y_i - m_i x_i$ and $b_{i-1} = y_i - m_{i-1} x_i$ the intercept term can be eliminated by introducing a new function $F(x, y, m) = \Omega(x, m, y - mx)$, and the final form of the solution thereby obtained is

$$\boxed{I = \frac{M}{2\pi} \sum_{i=1}^{n} F(x_i, y_i, m_{i-1}) - F(x_i, y_i, m_i).} \tag{8}$$

The function F is

$$F(x, y, m) = Ax \arctan(Ay) + C(y - mx) \arctan\left[C(x + my)\right] \tag{9}$$

where

$$A = \frac{1}{\sqrt{1 + x^2}}, \qquad C = \frac{1}{\sqrt{1 + m^2 + (y - mx)^2}}. \tag{10}$$

Equations (8), (9) and (10) provide a formula analogous to Lambert's formula for the irradiance due to a uniformly emitting polygon. The first term in (9) is independent of m, and therefore appears to cancel in the summand of (8) so it is tempting to omit it from F. But recall that terms of F with undefined m are omitted outright, so in the case where only one of m_i and m_{i-1} is undefined, there is nothing to cancel the first term. The terms do cancel if neither incident edge is vertical.

2.2 Remarks

There are several notable points about the result. Most importantly, the formula is a summation over a function of the vertices and the incoming and outgoing slopes m_{i-1} and m_i, respectively, and consequently may be evaluated in any order. In the case of an extraneous vertex, which has the same incoming and outgoing slope, the two F terms cancel and there is no contribution to the sum. Although the formula for F looks complicated, it is fairly easy to evaluate. Both the square root and arctangent functions have desirable computational behavior; note the radicand is bounded above 1.

The formula is valid only for a polygon which lies strictly above the plane of the receiver. As with Lambert's formula, the polygon must be clipped against the receiver plane, but unlike Lambert's formula, the projected polygon must be bounded on the image plane. (Otherwise the foregoing computation would have to be evaluated in the real projective plane, and this is the subject of Section 6.)

2.3 Error Bounds for the Image Plane

As noted above, the vertex-based formulation of the image plane integral requires a bounded projected polygon. If the polygon impinges the receiver plane the projection on the image plane will be unbounded. In this section we derive a simple bound on the error incurred by clipping the projected polygon on the image plane against a disc or square centered at the origin.

As it happens, the integral of (6) becomes simpler in polar coordinates, so we start by developing a bound in terms of a disc in the image plane. Let A be the complement of the open disc of radius R, centered at the origin. Formally

$$A = \{(x, y) : \sqrt{x^2 + y^2} \geq R\}.$$

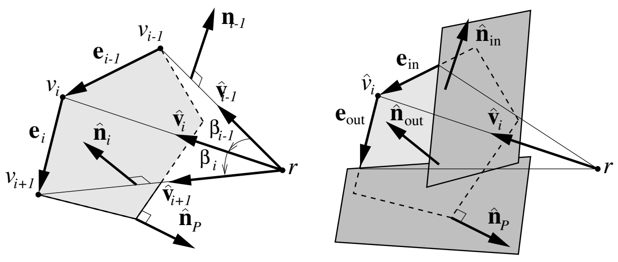


Figure 4: (a) *The vertex, edge, and edge normal vectors associated with vertex v_i.* (b) *When P is partially occluded but v_i remains visible, the incoming and outgoing edge vectors change only in magnitude, and the (unit) edge normals remain the same.*

If all of A is emits radiance uniformly with emission constant M, the resulting irradiance is computed from

$$\frac{M}{\pi}\int\!\!\int_A \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \frac{M}{\pi}\int_R^\infty\int_0^{2\pi}\frac{r}{(1+r^2)^2}\,d\theta\,dr = 2M\int_{1+R^2}^\infty \frac{1}{u^2}\,du = \frac{M}{1+R^2}.$$

This value therefore bounds the irradiance of any object with emission M which lies totally outside the disc of radius R. The exterior of the square with side-length $2R$ satisfies this and we have proved the following:

Lemma 1 *The error incurred in computing the irradiance from a uniformly emitting polygon with emission M by clipping inside the square with side length $2R$ centered at the origin on the image plane is bounded by $M/(1+R^2)$.*

Note $M/(1+R^2) < M/R^2$, so the bound is actually slightly better than the inverse square of the clipping square. In practice it might make more sense to clip the polygon in space against the viewing frustrum through the clipping square before projecting.

3 Vertex-Based Reformulation of Lambert's Formula

The goal of this section is to develop a vertex-based formulation of Lambert's formula analogous to the formula of the previous section, but based on the original polygon vertices rather than the image plane projection. Each term of the summation must depend only on the vertex and the local behavior of the incident edges, and hence can be evaluated in any order. For this, a formalization of what is meant by the "local behavior" at each vertex is required.

Define the *vertex vectors* of polygon P by $\mathbf{v}_i \equiv v_i - r$, and the *edge vectors* as $\mathbf{e}_i \equiv v_{i+1} - v_i$ and designate the *incoming* and *outgoing* edges at vertex v_i as the vectors $\mathbf{e}_{i-1}$ and $\mathbf{e}_i$, respectively. Each non-degenerate edge is contained in a unique plane containing the edge and the receiver point r. These edge planes contain the faces of the cone subtended by P from r, and therefore have an associated outward normal. We define the unit *outward edge normals*

$$\hat{\mathbf{n}}_i \equiv \frac{(v_i - r)\times \mathbf{e}_i}{\|(v_i - r)\times \mathbf{e}_i\|} = \frac{\mathbf{v}_i \times \mathbf{v}_{i+1}}{\|\mathbf{v}_i \times \mathbf{v}_{i+1}\|}.$$

Each $\hat{\mathbf{n}}_i$ is the outward normal of face i, containing edge i, of the cone subtended by P from r, as illustrated in Figure 4(a). We shall refer to $\hat{\mathbf{n}}_{i-1}$ and $\hat{\mathbf{n}}_i$ as the incoming and outgoing edge normals at vertex v_i.

If polygon P is partially occluded as viewed from r, but some neighborhood of vertex v_i remains visible, the incoming and outgoing edge vectors for the vertex corresponding to v_i in the clipped polygon do not change direction. Consequently, the incoming and outgoing outward normals are invariant, as illustrated in Figure 4(b).

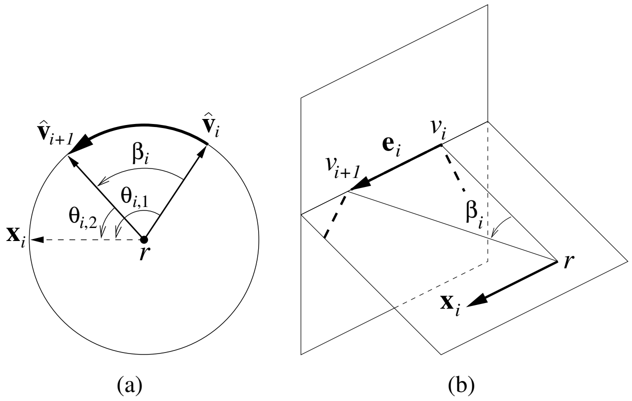


Figure 5: (a) *The angle β_i can be computed by subtracting the angles $\hat{\mathbf{v}}_{i+1}$ and $\hat{\mathbf{v}}_i$ make with a reference vector $\mathbf{x}_i$ (in the plane of edge i).* (b) *A natural choice for the reference vector is the edge vector $\mathbf{e}_i$ itself.*

3.1 Arc Length as an Angular Difference

Each edge plane intersects the unit sphere centered at r in the arc of a great circle. The arc is the edge projected onto the sphere, and the great circle is the intersection of the sphere with the edge plane. The length of the arc (the measure of the angle subtended by edge i from r) is β_i in (2) and is what makes the summation dependent on the edges of the polygon, as the natural way of computing β_i is by computing the inverse cosine of the inner product $\hat{\mathbf{v}}_i \cdot \hat{\mathbf{v}}_{i+1}$, as is done in (3).

The edge dependence can be avoided by expressing β_i as the *difference* of the angles that $\hat{\mathbf{v}}_i$ and $\hat{\mathbf{v}}_{i+1}$ make with some reference vector $\mathbf{x}_i$ in edge plane i. A choice for the reference vector is the normalized edge vector $\hat{\mathbf{e}}_i$ itself, as shown in Figure 5(b). In symbols, set

$$\begin{aligned} \theta_{i,1} &= \arccos(\mathbf{x} \cdot \hat{\mathbf{v}}_i) \\ \theta_{i,2} &= \arccos(\mathbf{x} \cdot \hat{\mathbf{v}}_{i+1}), \end{aligned}$$

so that $\beta_i = \theta_{i,1} - \theta_{i,2}$. Then

$$\beta_i = \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_i) - \arccos(\hat{\mathbf{v}}_{i+1} \cdot \hat{\mathbf{e}}_i) \tag{11}$$

where the unit edge vector $\hat{\mathbf{e}}_i$ is

$$\hat{\mathbf{e}}_i = \frac{\mathbf{e}_i}{\|\mathbf{e}_i\|} = \frac{v_{i+1} - v_i}{\|v_{i+1} - v_i\|}.$$

3.2 Formula

Equation 3 can be reformulated in terms of the vertices and the local behavior of the incident edges. Substituting equation (11) for β_i and rearranging produces

$$\begin{aligned} -I(r) &= \sum_{i=1}^{n} \left[\arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_i) - \arccos(\hat{\mathbf{v}}_{i+1} \cdot \hat{\mathbf{e}}_i)\right] \mathbf{N} \cdot \hat{\mathbf{n}}_i \\ &= \arccos(\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{e}}_1)\mathbf{N} \cdot \hat{\mathbf{n}}_1 - \arccos(\hat{\mathbf{v}}_2 \cdot \hat{\mathbf{e}}_1)\mathbf{N} \cdot \hat{\mathbf{n}}_1 + - \cdots + \arccos(\hat{\mathbf{v}}_n \cdot \hat{\mathbf{e}}_n)\mathbf{N} \cdot \hat{\mathbf{n}}_n - \arccos(\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{e}}_n)\mathbf{N} \cdot \hat{\mathbf{n}}_n \\ &= \sum_{i=1}^{n} \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_i)\, \mathbf{N} \cdot \hat{\mathbf{n}}_i - \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_{i-1})\, \mathbf{N} \cdot \hat{\mathbf{n}}_{i-1} \end{aligned}$$

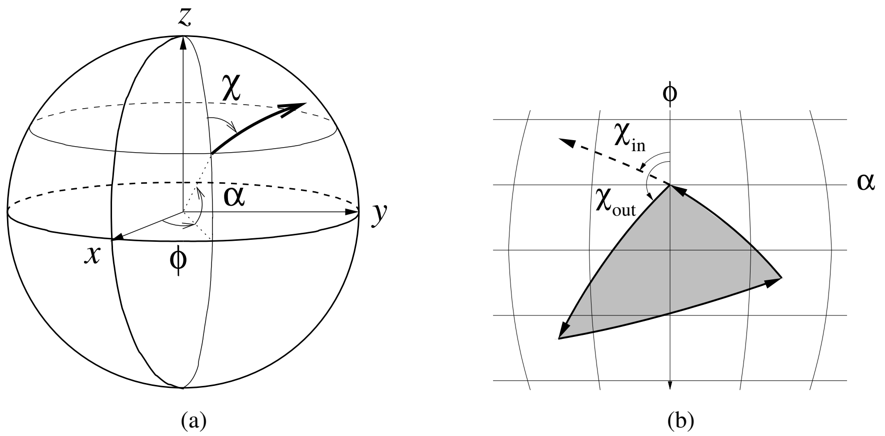


Figure 6: (a) *A directed arc on the sphere. The position of the tail of the arc has longitude ϕ, latitude α, and the position angle of the direction of the edge is χ (measured counter-clockwise from the meridian with respect to the inside of the sphere.)* (b) *A spherical polygon, viewed from the inside of the sphere. A vertex is characterized by its longitude and latitude $[\phi, \alpha]$ and the position angles χ_{in} and χ_{out} of the incoming and outgoing edges, respectively.*

Switching the terms of the summand gets rid of the negative of the sum, and thereby we arrive at the new formulation of Lambert's formula:

$$I(r) = \frac{M}{2\pi} \sum_{i=1}^{n} \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_{i-1})\, \mathbf{N} \cdot \hat{\mathbf{n}}_{i-1} - \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_i)\, \mathbf{N} \cdot \hat{\mathbf{n}}_i \tag{12}$$

$$= \frac{M}{\pi} \sum_{i=1}^{n} \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_{i-1})\, \mathbf{N} \cdot \frac{\mathbf{v} \times \mathbf{e}_{i-1}}{\|\mathbf{v} \times \mathbf{e}_{i-1}\|} - \arccos(\hat{\mathbf{v}}_i \cdot \hat{\mathbf{e}}_i)\, \mathbf{N} \cdot \frac{\mathbf{v} \times \mathbf{e}_i}{\|\mathbf{v} \times \mathbf{e}_i\|} \tag{13}$$

Equation (13) has the edge normal vectors replaced by their values in terms of the incoming and outgoing unit edge vectors.

Equations (12) and (13) are summations over the vertices of the polygon rather than the edges. The edge dependence of the original formula has been removed by using the edge vectors and normals.

4 Spherical Polygons

In this section we develop a vertex-based irradiance formula for a polygon on the sphere. Loosely, a spherical polygon is a region on the sphere which is bounded by a collection of great-circle arcs, the edges of the polygon. In terms of irradiance, we are only interested in spherical polygons contained in the hemisphere above the receiver plane.

While every polygon projects to a spherical polygon, not every spherical polygon is the projection of a planar polygon. As a trivial example, the hemisphere itself is a polygon with a single edge, consisting of an entire circle.

4.1 Spherical geometry

A brief review of spherical geometry is in order. As the basic elements of Euclidean geometry are points and lines in Euclidean space, the elements of spherical geometry are points and great circles on the sphere. A *great circle* is the intersection of the sphere and plane through the origin, the *poles* of the great circle are the points on the sphere which intersect the line through the center perpendicular to the plane. Informally, an *orientation* on a great circle is a direction for traversal; each great circle has two possible orientations. An orientation can be thought of as a choice of one of the poles, in such a way that the orientation of the great circle is "right-handed". This chosen pole is in fact the intersection of the *outward* normal to the plane of the great circle, and will hereafter be referred to as the *upper pole* of the great circle.

A connected portion of a great circle is a *great circle arc*, or, in this context, simply an *arc*. A *directed arc* is an arc with an orientation, and can be specified by ordering the endpoints. Notice that simply specifying two points on the sphere does not uniquely determine an arc, although it does determine a unique great circle. In fact, an oriented great circle is uniquely determined by a single pole, just as each point on the sphere is the upper pole of a unique oriented great circle.

4.2 Spherical coordinates

A coordinate system on the sphere is required. Spherical coordinates in the literature differ somewhat, depending on how they are used. We will use a system analogous to the longitude/latitude system, with longitudes measured positively to the east, and latitudes measured positively to the north as shown in Figure 6. Formally, we assume the sphere is a unit sphere in $\mathbf{R}^3$ placed at the origin with the north pole in the direction of the positive z-axis and the prime meridian in the x-positive portion of the xz-plane. The spherical longitude ϕ and latitude α (or azimuth and altitude) are related to the rectangular coordinates according to the equations

$$\begin{aligned} x &= \cos\phi\cos\alpha \\ y &= \sin\phi\cos\alpha \\ z &= \sin\alpha. \end{aligned}$$

The intersection of the sphere with the xy-plane is the *equator*, and is taken to have the usual orientation, making the upper pole the point $(0, 0, 1)$, the *north pole* of the sphere. The great semi-circles of constant longitude are the *meridians*, and the meridian with zero longitude is the *prime meridian*. The pole of the meridian with longitude ϕ is taken as the spherical point $[\phi - \pi/2, 0]$.

A more standard practice in graphics is to measure the latitude angle from the north pole rather than from the equator, probably because that corresponds to the angle a point, or ray, makes with a surface normal. This angle is often called the *colatitude* in cartography and astronomy, and we will use θ for this angle, for consistency with the normal angle, noting the relation $\theta = \pi/2 - \alpha$. [1] In either case, spherical coordinates are unique except for the north pole, and its diametric opposite the *south pole*, where longitude is undefined.

4.3 Spherical Polygons and Position Angles

A spherical polygon is the analogue of a planar polygon; the boundary instead consists of a finite collection of great circle arcs, the *edges* of the spherical polygon. An oriented spherical polygon consists of oriented edges in a natural way. We assume the spherical polygon has a *positive* orientation, which amounts to a counter-clockwise vertex order with respect to the center of the sphere.

In order to develop a formula for irradiance from spherical polygons in terms of local behavior, a local behavior characterization is required. Analogous to the incoming and outgoing slope of edges on the image plane, the formula will be in terms of incoming and outgoing *position angle*, which we now define.

The angle between two great circles, or great circle arcs, is the angle made by the planes which contain them. The *position angle* of an arc at a point on the sphere is the angle the (directed) arc makes with the

[1] The longitude/colatitude system of spherical coordinates is the one adopted my most calculus texts, but they tend to use θ for the longitude and ϕ for the colatitude—the opposite of use of the symbols commonly used in graphics.

(directed) meridian (Figure 6). The position angle of a directed arc is uniquely defined for any point on the arc except one of the poles, where there is no unique meridian. (In this case, the angle is usually measured against the prime meridian.) Position angles are full-circle angles, in the range $[0, 2\pi)$, as they are angles between directed arcs.

Each vertex of a spherical polygon is a point common to two edges. The *vertex angle* is the angle made by the two directed edges. The local vertex characterization, analogous to the incoming/outgoing slope formulation in Section 2, is in terms of the vertex position in spherical coordinates, and the position angles of the incoming and outgoing edges (Figure 6). The goal is to develop a vertex-based formula for irradiance from a spherical polygon in terms of vertex position and the two position angles.

For irradiance on a real surface we are only interested in polygons contained in the northern hemisphere, such as those that might arise from space polygons clipped against the receiver plane then projected onto the sphere. It is important to note, however, that not every spherical polygon contained in the northern hemisphere arises from such a projected space polygon. As a trivial example, the spherical polygon consisting the entire upper hemisphere cannot arise from a real (bounded) space polygon, nor can any spherical polygon with an edge of length π.

There is an important restriction on the edges of upper hemisphere polygons. Only half of the great circle which contains a non-equatorial edge lies above or on the xy-plane, so the maximum length of a non-equatorial edge is π. The points at which the great circle intersects the xy-plane are the *nodes* of the great circle; for an oriented great circle the nodes are identified as *ascending* and *descending nodes* in the natural way. As we shall see, the ascending node will serve as the reference point for measuring the arc length of an edge.

4.4 Projecting Polygons onto the Sphere

In order to develop a formula based on vertex position in spherical coordinates and position angles, we must relate the positions and position angles of a projected space polygon to the vertices of the space polygon. Suppose we have a space polygon P. The projection onto the unit sphere requires the rotation and translation portion of the transformation of (4) which translates the receiver point to $(0, 0, 0)$ and rotates so that the receiver normal is along the positive z-axis. The spherical positions of the projected vertices are then obtained from the definition of spherical coordinates. However, computing the position angles of the projected edges is more involved.

Note that the position angle of an edge varies along the edge so it is dependent on the spherical position on the edge. In other words, the position angle "of an edge" does not make sense unless it refers to the position angle of the edge at a particular point. We begin by relating the normal to the plane containing a spherical edge with a point on the edge and the position angle. Edges of a spherical polygon are directed, and as such they are directed circular arcs in space. Consider an edge through a spherical point $[\phi, \alpha]$ with position angle χ. The edge has a unique unit normal vector $\hat{\mathbf{n}}$ as previously described. The edge at the point can be viewed as a rotation of the equator, in the positive direction, passing through the point $[0, 0]$. The rotation which effects this (in the natural space coordinate system described above) is the matrix

$$R \equiv R_z(\phi) R_y(\alpha) R_x(\frac{\pi}{2} - \chi) \tag{14}$$

Using an auxiliary angle $\psi = \pi/2 - \chi$, the matrix R can be written explicitly

$$R = \begin{bmatrix} \cos\phi\cos\alpha & -\cos\phi\sin\alpha\sin\psi - \sin\phi\cos\psi & -\cos\phi\sin\alpha\cos\psi + \sin\phi\sin\psi \\ \sin\phi\cos\alpha & -\sin\phi\sin\alpha\sin\psi + \cos\phi\cos\psi & -\sin\phi\sin\alpha\cos\psi - \cos\phi\sin\psi \\ \sin\alpha & \cos\alpha\sin\psi & \cos\alpha\cos\psi \end{bmatrix} \tag{15}$$

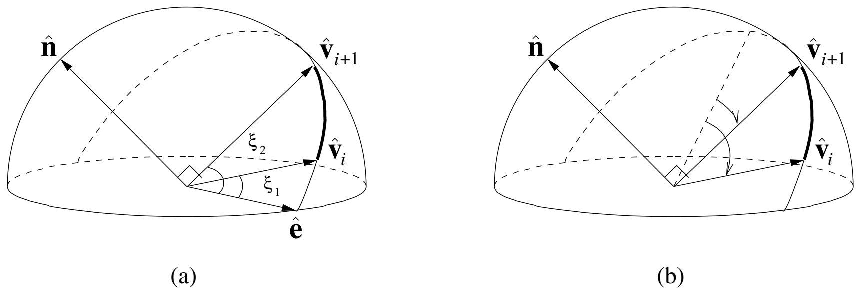


Figure 7: *Measuring the angle subtended by a spherical arc. (a) The angle as the difference of the angles the vertex vectors make with the vector* $\hat{\mathbf{e}}$ *pointing the toe ascending node. (b) The angle as the difference of the angles measured from the acme of the arc.*

The unit normal $\hat{\mathbf{n}}$ to the edge is then simply the unit normal to the positively-oriented equator, the vector $(0, 0, 1)$, transformed by the rotation matrix R:

$$\hat{\mathbf{n}} = R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\cos\phi \sin\alpha \cos\psi + \sin\phi \sin\psi \\ -\sin\phi \sin\alpha \cos\psi - \cos\phi \sin\psi \\ \cos\alpha \cos\psi \end{bmatrix} = \begin{bmatrix} -\cos\phi \sin\alpha \sin\chi + \sin\phi \cos\chi \\ -\sin\phi \sin\alpha \sin\chi - \cos\phi \cos\chi \\ \cos\alpha \sin\chi \end{bmatrix} \tag{16}$$

Equation (16) is of fundamental importance in this section, because it provides the relation of the edge normal to the position and orientation of a spherical edge at a point.

The position angle of an edge passing through a non-polar point (ϕ, α) is uniquely determined:

$$\sin\chi = -n_x \sin\alpha \cos\phi - n_y \sin\alpha \sin\phi + n_z \cos\alpha \tag{17}$$

$$\cos\chi = n_x \sin\phi - n_y \cos\phi \tag{18}$$

From the z-coordinate of (16) we also have

$$\sin\chi = \frac{n_z}{\cos\alpha} \tag{19}$$

which is undefined for at the north pole ($\alpha = \pi/2$), but so is the position angle. (Edges through the north pole of the sphere have no contribution, because the edge normal is perpendicular to the receiver normal.) In any case, these expressions together with the definition of spherical coordinates, provide a formula for converting the position and edge normals of each vertex of a polygon P to a spherical position and position angles of the adjacent edges.

4.5 Lambert's Formula on the Sphere

Lambert's formula states that the contribution of each spherical edge is the size (arc length) of the edge times the cosine of the angle the plane of the edge makes with the normal. The latter is easily obtained from the unit normal $\hat{\mathbf{n}}$ of the edge; it is the negative of the inner product of $\hat{\mathbf{n}}$ with the unit z-axis, and this is simply the z-coordinate n_z from (16)

$$\cos\gamma = -n_z = -\cos\alpha \sin\chi. \tag{20}$$

The arc length β is more difficult. The goal is to measure the arc lengths from each of the two vertices to a *fiducial point* on the edge and express the length of the edge as the difference of the two lengths.

Consider first the case of a non-equatorial edge. As the polygon is restricted to the upper hemisphere, the non-equatorial edge is necessarily contained in a great semi-circle which has an ascending node. The direction of the ascending node, in rectangular coordinates, is given by the cross product of the z-axis with the edge normal $\hat{\mathbf{n}}$. In symbols

$$\mathbf{e} \equiv (0,0,1) \times \hat{\mathbf{n}} = \begin{bmatrix} \sin\phi \sin\alpha \cos\psi + \cos\phi \sin\psi \\ -\cos\phi \sin\alpha \cos\psi + \sin\phi \sin\psi \\ 0 \end{bmatrix} \tag{21}$$

and the unit vector is

$$\hat{\mathbf{e}} = \frac{1}{\sqrt{\sin^2\alpha \cos^2\psi + \sin^2\psi}} \begin{bmatrix} \sin\phi \sin\alpha \cos\psi + \cos\phi \sin\psi \\ -\cos\phi \sin\alpha \cos\psi + \sin\phi \sin\psi \\ 0 \end{bmatrix}. \tag{22}$$

The angle ξ from the ascending node to a point (ϕ, α), in rectangular coordinates $\hat{\mathbf{v}}$, satisfies, with some simplification,

$$\cos\xi = \hat{\mathbf{v}} \cdot \hat{\mathbf{e}} = \frac{\cos\alpha \sin\psi}{\sqrt{\sin^2\alpha \cos^2\psi + \sin^2\psi}}. \tag{23}$$

Consequently, the angular size of the edge joining $[\phi_i, \alpha_i]$ and $[\phi_{i+1}, \alpha_{i+1}]$, with position angles χ_i, χ_{i+1}, respectively, is the difference

$$\beta_i = \xi_{i,2} - \xi_{i,1} = \arccos \frac{\cos\alpha_{i+1} \sin\psi_{i+1}}{\sqrt{\sin^2\alpha_{i+1} \sin^2\psi_{i+1} + \sin^2\psi_{i+1}}} - \arccos \frac{\cos\alpha_i \sin\psi_i}{\sqrt{\sin^2\alpha_i \sin^2\psi_i + \sin^2\psi_i}}. \tag{24}$$

A somewhat simpler alternative formulation is also possible in terms of inverse tangents, obtainable by measuring from the acme of the arc (the point midway between the two nodes, as shown in Figure 7)

$$\begin{aligned} \beta_i &= \arctan \frac{\cos\alpha_i \sin\psi_i}{\sin\alpha_i} - \arctan \frac{\cos\alpha_{i+1} \sin\psi_{i+1}}{\sin\alpha_{i+1}} && (25) \\ &= \arctan \frac{\cos\chi_i}{\tan\alpha_i} - \arctan \frac{\cos\chi_{i+1}}{\tan\alpha_{i+1}}. && (26) \end{aligned}$$

In this latter formulation, the arctangent must be extended to include $-\pi/2$ and $\pi/2$ for a zero denominator with a negative and positive numerator. A zero numerator and denominator is not possible for a non-equatorial edge.

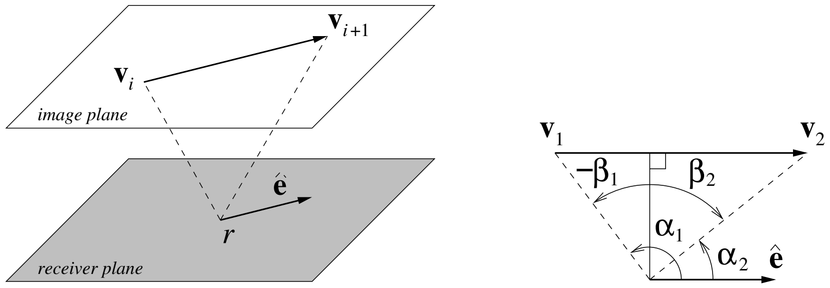
The case of an equatorial edge is conceptually easier, because we have $\cos\gamma = 1$, and the arc lengths can be measured by subtracting longitudes. However, a problem occurs if an edge crosses the prime meridian as the difference will be off by -2π. It seems the only direct solution to this problem is to "cut" equatorial edges at the prime meridian by adding a correction term to the formula.

Summing the edge terms, and rearranging in the manner of the previous sections, we can write the irradiance directly in terms of the vertex positions and incident position angles

$$I(r) = cM + \frac{M}{2\pi} \sum_{i=1}^{n} G(\phi_i, \alpha_i, \chi_{i-1}) - G(\phi_i, \alpha_i, \chi_i) \tag{27}$$

where

$$G(\phi, \alpha, \chi) = \begin{cases} 0, & \text{if } \alpha = \pi/2 \\ \phi, & \text{if } \alpha = 0 \text{ and } \chi = \pi/2 \\ -\phi, & \text{if } \alpha = 0 \text{ and } \chi = 3\pi/2 \\ \cos\alpha \sin\chi \arctan \dfrac{\cos\chi}{\tan\alpha}, & \text{otherwise.} \end{cases} \tag{28}$$



(a) (b)

Figure 8: (a) *The plane of an edge on the image plane, and the unit edge vector* $\hat{\mathbf{e}}$. (b) *Measuring the angles subtended of an edge (in the plane of the edge) from the edge vector and from the normal to the edge.*

and c is the number of equatorial edges which properly cross the prime meridian (0 or 1, for a proper spherical polygon.) The correction could, of course, be avoided if we measure the angles of equatorial edges from the edge vector in space, as was done in Section 3, but doing so would require the original space vertices of the polygon.

5 The Image Plane Revisited

In this section we return to the situation of the first section, where a polygon (or an entire scene) is projected onto the image plane. We ultimately wish to address points and edges "at infinity" which correspond to points and edges along the equator of the hemisphere. The subject of this section is to relate the image plane formula (8) to Lambert's formula and therefore to the sphere formula of the previous section. Doing so will clarify how the image plane formula can be augmented to handle vertices corresponding to equatorial points on the sphere. We begin by demonstrating how the image plane formula can be extracted directly from Lambert's formula without using a separate Green's theorem integration.

On the image plane, a (finite) polygonal *edge* is simply a line segment, which for definiteness we take as the segment joining $(x_i, y_i, 1)$ and $(x_{i+1}, y_{i+1}, 1)$. For now, assume the edge is not vertical and therefore is contained in the line $y = m_i x + b_i$. In Lambert's formula, the irradiance contribution of an edge is the apparent angular size β of the edge times the cosine of the angle γ the edge plane normal makes with the surface normal. Both of these angles are measured from the point of evaluation r on the receiver, which in image plane coordinates is simply $(0, 0, 0)$.

We attack the cosine of the plane angle first. In image plane coordinates, the edge is the difference $(x_i, y_i, 1)$–$(x_{i+1}, y_{i+1}, 1)$, and the outward edge plane normal is $\mathbf{n}_i = (x_i, y_i, 1) \times (x_{i+1}, y_{i+1}, 1)$. The cosine is then

$$\cos\gamma_i = -(0,0,1)\cdot\mathbf{n}_i = \frac{-x_i y_{i+1} + y_i x_{i+1}}{\sqrt{(y_i - y_{i+1})^2 + (x_{i+1} - x_i)^2 + (x_i y_{i+1} - y_i x_{i+1})^2}} \tag{29}$$

To remove the dependence on the second endpoint, let $t = x_{i+1} - x_i$, so that $x_{i+1} = x_i + t$ and $y_{i+1} = y_i + m_i t$. Then the expression for $\cos\gamma_i$ becomes

$$\cos\gamma_i = \frac{-x_i(y_i + m_i t) + y_i(x_i + t)}{\sqrt{(y_i - (y_i - m_i t))^2 + (x_i + t - x_i)^2 + (x_i(y_i + m_i t) + y_i(x_i + t))^2}} \tag{30}$$

$$= \frac{t(y_i - m_i x_i)}{\sqrt{t^2 m_i^2 + t^2 + t^2(m_i x_i - y_i)^2}} \tag{31}$$

$$= \frac{s(y_i - m_i x_i)}{\sqrt{1 + m_i^2 + (m_i x_i - y_i)^2}}, \tag{32}$$

where $s = \mathrm{sgn}(t)$ is the direction, of sorts, of the edge on the image plane.

The same trick used to compute the angle β by measuring from the edge vector (see Section 3) can be used here. The edge vector, as illustrated in Figure 8, is

$$\hat{\mathbf{e}} = \frac{s}{\sqrt{1 + m_i^2}}(1, m_i, 0) \tag{33}$$

and the vertex vectors are

$$\hat{\mathbf{v}}_1 = \frac{(x_i, y_i, 1)}{\sqrt{1 + x_i + y_i}}, \hat{\mathbf{v}}_2 = \frac{(x_{i+1}, y_{i+1}, 1)}{\sqrt{1 + x_{i+1} + y_{i+1}}} \tag{34}$$

The angle of the edge is the difference of the vertex angles measured from the edge vector. We have, for $\hat{\mathbf{v}}_1$,

$$\cos\alpha_1 = \hat{\mathbf{e}} \cdot \hat{\mathbf{v}}_1 = \frac{s(x_i + m_i y_i)}{\sqrt{(1 + m_i^2)(1 + x_i^2 + y_i^2)}}. \tag{35}$$

However, the image plane formula involves inverse tangents rather than inverse cosines. To match the formulation, we use the following identity for first-quadrant angles

$$\arccos\frac{a}{c} = \arctan\frac{\sqrt{c^2 - a^2}}{a} = \frac{\pi}{2} - \arctan\frac{a}{\sqrt{c^2 - a^2}} \tag{36}$$

and thereby obtain

$$\alpha_1 = \frac{\pi}{2} - \arctan\frac{s(x_i + m_i y_i)}{\sqrt{1 + m_i^2 + (y_i - m_i x_i)^2}} \tag{37}$$

Using $\beta_1 = \pi/2 - \alpha_1$, the angle of $\hat{\mathbf{v}}_1$ measured from the perpendicular in the edge plane (as illustrated in Figure 8, and corresponding to measuring from the acme of the arc in the previous section) the arctangent conversion remains valid if the numerator $x_i + m_i y_i$ becomes negative; the angle β_1 likewise becomes negative, as it is measured. Then the angle of the edge β_i is the difference $\beta_2 - \beta_1$ (β_2 computed analogously to β_1) and we obtain

$$\beta_i = s\left(\arctan\frac{x_{i+1} + m_{i+1} y_{i+1}}{\sqrt{1 + m_{i+1}^2 + (y_{i+1} - m_{i+1} x_{i+1})^2}} - \arctan\frac{x_i + m_i y_i}{\sqrt{1 + m_i^2 + (y_i - m_i x_i)^2}}\right). \tag{38}$$

The term for the edge is the product $\beta_i \cos\gamma_i$ in which the sign terms cancel ($s^2 = 1$) and we have

$$\left(\arctan\frac{x_{i+1} + m_{i+1} y_{i+1}}{\sqrt{1 + m_{i+1}^2 + (y_{i+1} - m_{i+1} x_{i+1})^2}} - \arctan\frac{x_i + m_i y_i}{\sqrt{1 + m_i^2 + (y_i - m_i x_i)^2}}\right) \frac{(y_i - m_i x_i)}{\sqrt{1 + m_i^2 + (m_i x_i - y_i)^2}} \tag{39}$$

which matches the formula developed from Green's theorem—but where is the second term? Recall the development in the section thus far has assumed that the the edge on the image plane is not vertical.

5.1 Vertical edges

Suppose now that the edge joining $(x_i, y_i, 1)$ and $(x_{i+1}, y_{i+1}, 1)$ is vertical. In this case, the line equation is simply $x = x_i$ or $x = x_{i+1}$ and the expression for $\cos\gamma_i$ reduces to

$$\cos\gamma_i = -\frac{sx_i}{\sqrt{1+x_i^2}} \tag{40}$$

where $s = \operatorname{sgn}(y_{i+1} - y_i)$, and the edge angle becomes

$$\beta_i = s\left(\arctan\frac{y_{i+1}}{\sqrt{1+x_{i+1}^2}} - \arctan\frac{y_i}{\sqrt{1+x_i^2}}\right). \tag{41}$$

Again the s signs cancel in the product.

While equation (41) matches the first term in the expression for the F in the Green's Theorem evaluation, there is a problem, because that summation *excluded* vertical edges. Yet the previous equation shows that vertical edges have a definite contribution *via* Lambert's summation. This can be reconciled by adding the terms

$$\frac{x_i}{\sqrt{1+x_i^2}}\arctan\frac{y_i}{\sqrt{1+x_i^2}} - \frac{x_{i+1}}{\sqrt{1+x_{i+1}^2}}\arctan\frac{y_{i+1}}{\sqrt{1+x_{i+1}^2}} \tag{42}$$

to the Lambert term for each non-vertical edge. The terms telescope for adjacent non-vertical edges, so if the summation excludes vertical edges the required terms for the vertical edges will remain in the summation. It is left to the reader to verify that this does indeed produce exactly the same formula as the one developed directly from Green's Theorem.

5.2 Remarks

The results in this section show how the terms of the image plane formula correspond to the equivalent terms which are obtained from Lambert's formula. This correspondence will be exploited in the next section, to merge the expressions obtained from the sphere formula to vertices and edges "at infinity" in the image plane.

6 The Projective Plane

A restriction of the the image plane formula developed in Section 2 is that it requires the polygon to lie strictly above the receiver plane (the xy-plane), whereas Lambert's formula allows the polygon to have vertices or edges actually on the receiver plane. A polygon having a vertex or an edge on the receiver plane will project to an unbounded polygon on the image plane, as illustrated in Figure 9. When a single vertex lies on the receiver plane, the projected polygon has a vertex "at infinity"; when an edge lies on the receiver plane, the projected polygon has an edge "at infinity". In this section we shall augment the image plane formula to handle such unbounded polygons, by working in the real projective plane [8].

The projective plane arises naturally from the perspective projection used in Section 2. A point (x, y, z) in space with $z \neq 0$ projects to the point $(x/z, y/z, 1)$ on the image plane. Points on the xy-plane do not project to the image plane, rather they project to *points at infinity*.

In fact, the image plane we have used, the plane parallel to the xy-plane one unit above, matches the homogeneous representation of the projective plane. Points on the projective plane are either *finite* points, represented as $(x, y, 1)$ or *points at infinity*, which are represented as $(a, b, 0)$; any nonzero multiple of a point at infinity is the same point at infinity.

The projective plane is not a vector space, but a vector can be defined as the difference of any two finite points, thus having the form form $(a, b, 0)$. A vector represents a direction on the finite plane in a manner similar to a point at infinity, with one important difference. Without giving formal definitions, a vector and a point at infinity differ in that a negative multiple of a point at infinity results in the same point, while such a multiple reverses the direction of a vector. For this reason, we shall refer to the direction of a vector as a *signed direction*, and a point at infinity as a *projective direction*.

6.1 Reformulation on the Finite Plane

The vertex-based formulation works on the finite projective plane (which excludes points at infinity) in the same manner as in Section 2. However, it is more natural to replace the edge slopes with projective directions. A line with slope m, regardless of its position on the finite plane, intersects the unique point at infinity $(1, m, 0)$; a vertical line intersects $(0, 1, 0)$. The slope of a line may therefore be generally represented by the point at infinity $(a, b, 0)$ where, for a line with finite slope, $m = a/b$ and for a vertical line we use $a = 0$, $b = 1$.

For a non-vertical edge, replacing m with b/a in equation produces

$$\frac{ax + by}{\sqrt{a^2 + b^2 + (bx - ay)^2}} \arctan \frac{ay - bx}{\sqrt{a^2 + b^2 + (bx - ay)^2}} \tag{43}$$

which not only becomes more symmetric, but reduces immediately to equation (41) for a vertical edge. Using projective directions rather than the slopes of the incoming and outgoing edges thus results in a cleaner formulation. It is worth stating the formulation specifically:

$$I = \frac{M}{2\pi} \sum_{i=1}^{n} \tilde{F}(x_i, y_i, a_{i-1}, b_{i-1}) - \tilde{F}(x_i, y_i, a_i, b_i). \tag{44}$$

where

$$\tilde{F}(x, y, a, b) = \frac{ax + by}{\sqrt{a^2 + b^2 + (bx - ay)^2}} \arctan \frac{ay - bx}{\sqrt{a^2 + b^2 + (bx - ay)^2}} \tag{45}$$

and (a_i, b_i) is the projective direction of edge i. Equation (44) provides an alternative to the original image plane formula (8) that eliminates the extra condition for vertical edges. Also the projective directions of the edges are easy to compute: $(a_i, b_i) = (x_{i+1} - x_i, y_{i+1} - y_i)$, and any multiple of this direction, including a negative multiple, remains valid.

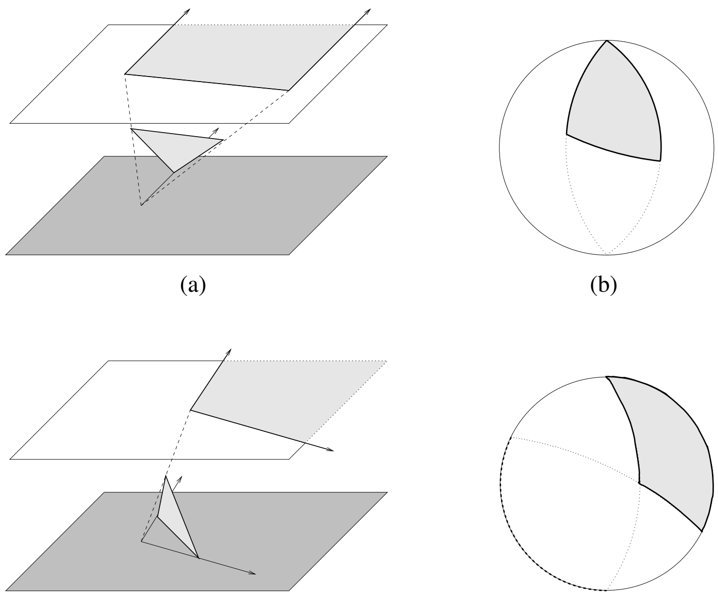
6.2 Vertices at Infinity

The primary goal is to develop a vertex-based formula which works on the polygon in the space in its natural representation without referring to the original polygon. In this case the natural representation of a projective polygon is it terms of projective points. At first glance, it might appear that because the projective direction of an edge going to infinity is identical to that point at infinity the formula would be simpler still, but this is not the case. As we shall see, the *signed* direction of the edges incident on an infinite vertex are needed for a vertex at infinity.

For a concrete example, consider the term for a vertical edge

$$\frac{x_i}{\sqrt{1 + x_i}} \left[\arctan \frac{y_{i+1}}{\sqrt{1 + x_{i+1}}} - \arctan \frac{y_i}{\sqrt{1 + x_i}} \right].$$

The first term approaches $\pi/2$ as y_{i+1} approaches infinity, but $-\pi/2$ as y_{i+1} approaches negative infinity. However, in both cases the vertex at infinity is represented by $(0, 1) \equiv (0, -1)$ in the projective plane, and



(a) (b) (c) (d)

Figure 9: (a) *A vertex that lies on the receiver plane projects to a vertex at infinity on the projective plane, resulting in an unbounded polygon with parallel edges approaching infinity.* (b) *The projected polygon in a disc representation of the projective plane. The edges are contained in lines which meet at diametrically opposite points; these two points together are the single point at infinity.* (c) *An edge on the receiver plane projects to an edge at infinity on the projective plane, resulting in two non-parallel edges which go to infinity.* (d) *The infinite edge represented in the projective plane.*

if only the vertex and the incoming direction is used to represent the vertex, there is no way to distinguish the sign of the $\pi/2$ term.

The local characterization of a vertex at infinity therefore requires a signed direction $(a, b, 0)$ (which without the direction is also the point at infinity on which the vertex lies) and an arbitrary point $(x_{\rm in}, y_{\rm in}, z_{\rm in})$ and $(x_{\rm out}, y_{\rm out}, z_{\rm out})$ on each incident edge. Either point may be a point at infinity. A finite vertex requires only a projective (unsigned) direction, and the position of the vertex itself provides the point on the edge.

Now consider the case of an edge terminating at a vertex at infinity. The characterization of the incoming edge at the vertex requires an arbitrary (finite) point on the edge $(x_{\rm in}, y_{\rm in}, 1)$ and a signed direction $(a, b, 0)$. As we have seen, the term for the incoming edge is the product of the $\cos\gamma$ term and the angle measured from the near point on the line containing the edge. The former is computed as in Equation (23), while the latter is the limiting value $\pi/2$—the sign is always positive for an incoming edge. For an outgoing edge, the cosine term is the same, but the angular term is always $-\pi/2$. Note that because of this the s terms from Section 5 do not cancel and it is therefore necessary to use a signed direction for the edge.

6.3 Edges at Infinity

The case of an edge at infinity corresponds to an edge on the equator of the sphere. The cosine of the edge angle is always 1, and the angular size can be computed in a manner analogous to the longitude subtraction which we did on the sphere. There is a topological problem, however; an infinite edge in the projective plane is not well represented by a circular arc, rather it is a pair of antipodal arcs. This is not a problem if we assume each infinite edge comes from a real polygonal edge, and thus cannot span more than a semi-circle in this representation. The angle of the edge is the difference of the angle each endpoint makes with the point at infinity $(0, 1, 0)$: the contribution of an incoming edge at infinity to $(x, y, 0)$ is the angle $\arctan(x/y)$, for an outgoing edge, it is the negative.

Again, we have the problem with an edge at infinity which crosses the point $(1, 0, 0)$; in this case an extra correction of π must be added. Again, if we measure the angles from the edge vector of the polygon the correction can be avoided; this is done in the next subsection.

Combining the expressions for vertices and edges at infinity with the finite edge formulations of the previous section we obtain a complete formula for polygons on the image plane, including unbounded polygons. All the terms are collected in Figure 6.4. Notice we have not attempted to unify the terms for the various cases into a single expression.

6.4 Formulation in Homogeneous Coordinates

The development in the previous subsection developed a vertex-based formula for irradiance from a polygon situated and represented in the real projective plane without referring to the original polygon. An argument could be made that the projective plane is not the proper setting for computing irradiance, because antipodal points at infinity are most definitely not the same point; the fact that the signed direction for points at infinity was needed exemplifies this. In this section we amend the formula so that it applies to the homogeneous coordinates of the projected polygon. The principal difference is that we omit points at infinity outright and simply use the coordinates of vertices which line on the receiver plane directly.

Consider the vertices of the projected polygon P^* in homogeneous image plane coordinates. We make the assumption that vertices which lie on the receiver plane, *i.e.* with z-coordinate 0, remain unaffected by the perspective projection. The polygon vertices are then of the form (x_i, y_i, z_i) where z_i is either 0 or 1.

Each edge has a direction vector. The edge joining two finite vertices has direction $(x_{i+1} - x_i, y_{i+1} - y_i, 0)$, the vector difference on the image plane. For the edge joining two vertices at infinity, *i.e.* two vertices on the receiver plane, we use the same vector. The direction of an edge going to a vertex at infinity is simply that vertex at infinity $(x_i, y_i, 0)$, and the negative for an edge leaving a vertex at infinity. Again, to properly specify the "local behavior" of an edge going to (or coming from) a vertex at infinity we require an arbitrary point on the edge.

We previously assumed that each vertex of the projective polygon is either a finite point or a point at infinity. That is, points at infinity did not have a direction. In the homogeneous representation, a vertex at infinity has the coordinates of the vertex on the receiver plane, and as such has a definite direction (that is, a negative multiple of the point produces a vertex in the opposite direction—again, on the projective plane these points are the same.) So rather than using a signed direction of the edge going to a point at infinity, we can use the homogeneous point itself.

To measure the angular span of an edge at infinity, we shall measure against the edge vector on the receiver plane—this eliminates the need for the π correction in the previous section. We have, as in Section 5

$$\begin{aligned} \theta_{i,1} &= \arccos \frac{(x_i, y_i, 0) \cdot (a_i, b_i, 0)}{\sqrt{x_i^2 + y_1^2}\sqrt{a_i^2 + b_i^2}} \\ &= \frac{\pi}{2} - \arctan \frac{a_i x_i + b_i y_i}{|b_i x_i - a_i y_i|} \end{aligned}$$

where $(x_i, y_i, 0)$ is either point on the edge and $(a_i, b_i, 0)$ is the direction of the edge. The term for the edge is therefore

$$\arctan \frac{a_i x_{i+1} + b_i y_{i+1}}{|b_i x_{i+1} - a_i y_{i+1}|} - \arctan \frac{a_i x_i + b_i y_i}{|b_i x_i - a_i y_i|} \tag{46}$$

This naturally leads to a vertex-based expression, where the edge vector $(a_i, b_i, 0)$ is replaced by the appropriate incoming or outgoing edge direction vector. Figure 6.4 contains a complete list and description of all the terms.

	incoming edge	outgoing edge
edge	$C(a_{\text{in}}y - b_{\text{in}}x)\arctan\left[C(a_{\text{in}}x + b_{\text{in}}y)\right]$	$-C(a_{\text{out}}y - b_{\text{out}}x)\arctan\left[C(a_{\text{out}}x + b_{\text{out}}y)\right]$
vertical edge	$Ax\arctan(Ay)$	$-Ax\arctan(Ay)$
vertex at infinity	$\frac{\pi}{2}\,C(a_{\text{in}}y_{\text{in}} - b_{\text{in}}x_{\text{in}})$	$-\frac{\pi}{2}\,C(a_{\text{out}}y_{\text{out}} - b_{\text{out}}x_{\text{out}})$
edge at infinity	$\arctan\frac{x}{y}$	$-\arctan\frac{x}{y}$

Where

$$A = \frac{1}{\sqrt{1+x^2}}, \qquad C = \frac{1}{\sqrt{a^2+b^2+(bx-ay)^2}}$$

The terms for the vertical edges are redundant but included for completeness. Each projective vertex is a finite vertex or point at infinity, each is of the form $(x, y, 0)$. Each incident edge has a signed direction $(a, b, 0)$. The x_{in} and y_{in} values in the terms for a vertex at infinity are arbitrary points on the incoming edge; the "out" values are those of the outgoing edge. A correction of π must be added if an edge at infinity crosses $(1, 0, 0)$. The irradiance is the sum of the terms and the correction scaled by $M/2\pi$.

Figure 10: Terms for the vertex-based irradiance formulated in the projective plane.

	incoming edge	outgoing edge
edge	$C(a_{\text{in}}y - b_{\text{in}}x)\arctan\left[C(a_{\text{in}}x + b_{\text{in}}y)\right]$	$-C(a_{\text{out}}y - b_{\text{out}}x)\arctan\left[C(a_{\text{out}}x + b_{\text{out}}y)\right]$
vertical edge	$Ax\arctan(Ay)$	$-Ax\arctan(Ay)$
vertex at infinity	$\frac{\pi}{2}\,C(xy_{\text{in}} - yx_{\text{in}})$	$-\frac{\pi}{2}\,C(xy_{\text{in}} - yx_{\text{out}})$
edge at infinity	$\arctan\frac{a_{\text{in}}x + b_{\text{in}}y}{\lvert b_{\text{in}}x - a_{\text{in}}y\rvert}$	$-\arctan\frac{a_{\text{out}}x + b_{\text{out}}y}{\lvert b_{\text{out}}x - a_{\text{out}}y\rvert}$

Where

$$A = \frac{1}{\sqrt{1+x^2}}, \qquad C = \frac{1}{\sqrt{a^2+b^2+(bx-ay)^2}}$$

The terms for the vertical edges are redundant but included for completeness. Each homogeneous vertex is a either finite vertex of the form $(x, y, 1)$ or a vertex at infinity of the form $(x, y, 0)$, in image plane coordinates. Each incident edge has an unsigned direction $(a, b, 0)$. The x_{in} and y_{in} values in the terms for a vertex at infinity are arbitrary points on the incoming edge; the "out" values are those of the outgoing edge. The irradiance is the sum of the terms scaled by $M/2\pi$.

Figure 11: Terms for the vertex-based irradiance formulated in homogeneous coordinates.

7 Conclusion

This report has detailed several vertex-based formulas for irradiance due to a uniformly emitting polygon. Each has advantages and disadvantages. The image plane formula from Section 2 is the simplest and perhaps the most computationally efficient, but suffers from the restriction that the polygon is not allowed to impinge the receiver plane. The reformulation of Lambert's formula given by Equation (12) works without transforming the polygon and has perhaps the cleanest formulation but does not immediately generalize to the apparent intersection of polygons. The sphere formula of Section 4 was introduced primarily to guide the extension of the image plane formula to unbounded polygons, and these formulas given in Section 6 have extra "ifs". We hope the vertex-based formulations for irradiance developed here will provide a useful alternative to Lambert's formula.

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